

# On Weyl modules of cyclotomic $q$ -Schur algebras

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**ABSTRACT.** We study on Weyl modules of cyclotomic  $q$ -Schur algebras. In particular, we give the character formula of the Weyl modules by using the Kostka numbers and some numbers which are computed by a generalization of Littlewood-Richardson rule. We also study corresponding symmetric functions. Finally, we give some simple applications to modular representations of cyclotomic  $q$ -Schur algebras.

## § 0. INTRODUCTION

Let  ${}_R\mathcal{H}_{n,r}$  be the Ariki-Koike algebra over a commutative ring  $R$  with parameters  $q, Q_1, \dots, Q_r \in R$  associated to the complex reflection group  $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ , and let  ${}_R\mathcal{S}_{n,r}$  be the cyclotomic  $q$ -Schur algebra associated to  ${}_R\mathcal{H}_{n,r}$  introduced by [DJM]. Put  $\mathcal{A} = \mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_r]$ , where  $q, Q_1, \dots, Q_r$  are indeterminate, and  $\mathcal{K} = \mathbb{Q}(q, Q_1, \dots, Q_r)$  is the quotient field of  $\mathcal{A}$  (In this introduction, we omit the subscript  $\mathcal{K}$  when we consider an algebra over  $\mathcal{K}$ ).

In the case where  $r = 1$ ,  ${}_R\mathcal{H}_{n,1}$  is the Iwahori-Hecke algebra associated to the symmetric group  $\mathfrak{S}_n$ , and  ${}_R\mathcal{S}_{n,1}$  is the  $q$ -Schur algebra associated to  ${}_R\mathcal{H}_{n,1}$ . In this case, the  $q$ -Schur algebra  ${}_R\mathcal{S}_{n,1}$  comes from the Schur-Weyl duality between  ${}_R\mathcal{H}_{n,1}$  and the quantum group  ${}_RU_q(\mathfrak{gl}_m)$  as follows. Let  $\mathfrak{gl}_m$  be the general linear Lie algebra, and  $U_q(\mathfrak{gl}_m)$  be the corresponding quantum group over  $\mathcal{K}$ . We consider the vector representation  $V$  of  $U_q(\mathfrak{gl}_m)$ , then  $U_q(\mathfrak{gl}_m)$  acts on the tensor space  $V^{\otimes n}$  via coproduct of  $U_q(\mathfrak{gl}_m)$ .  $\mathcal{H}_{n,1}$  also acts on the tensor space  $V^{\otimes n}$  by a  $q$ -analogue of the permutations for the ingredient of the tensor product. Then the Schur-Weyl duality between  $U_q(\mathfrak{gl}_m)$  and  $\mathcal{H}_{n,1}$  holds via this tensor space  $V^{\otimes n}$  by [J]. Moreover, this Schur-Weyl duality also holds even over  $\mathcal{A}$  (see [Du]). Hence, we can specialize to any ring  $R$  with a parameter  $q \in R^\times$ . Then the  $q$ -Schur algebra  ${}_R\mathcal{S}_{n,1}$  coincides with the image of  ${}_RU_q(\mathfrak{gl}_m) \rightarrow \text{End}(V^{\otimes n})$  which comes from the action of  ${}_RU_q(\mathfrak{gl}_m)$  on  $V^{\otimes n}$ .

On the other hand, in the case where  $r \geq 2$ , it is also known the Schur-Weyl duality by [SakS]. Let  $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \dots \oplus \mathfrak{gl}_{m_r}$  be a Levi subalgebra of  $\mathfrak{gl}_m$ , and  $U_q(\mathfrak{g})$  be the corresponding quantum group over  $\mathcal{K}$ .  $U_q(\mathfrak{g})$  acts on  $V^{\otimes n}$  by the restriction of the action of  $U_q(\mathfrak{gl}_m)$ . We can also define the action of  $\mathcal{H}_{n,r}$  on  $V^{\otimes n}$  which is a generalization of the action of  $\mathcal{H}_{n,1}$ . Then  $U_q(\mathfrak{g})$  and  $\mathcal{H}_{n,r}$  satisfy the Schur-Weyl duality via the tensor space  $V^{\otimes n}$  by [SakS]. Unfortunately, this Schur-Weyl duality does not hold over  $\mathcal{A}$  since the action of  $\mathcal{H}_{n,r}$  on  $V^{\otimes n}$  is not defined over  $\mathcal{A}$ . However, we can replace  $\mathcal{H}_{n,r}$  with the modified Ariki-Koike algebra  ${}_R\mathcal{H}_{n,r}^0$  over  $R$  with

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\*This research was supported by GCOE ‘Fostering top leaders in mathematics’, Kyoto University.

parameters  $q, Q_1, \dots, Q_r$  such that  $Q_i - Q_j$  ( $i \neq j$ ) is invertible in  $R$  which was introduced by [Sho]. Then, the Schur-Weyl duality between  ${}_R U_q(\mathfrak{g})$  and  ${}_R \mathcal{H}_{n,r}^0$  holds via the tensor space  $V^{\otimes n}$  (see [SawS]). Let  ${}_R \mathcal{S}_{n,r}^0$  be the image of  ${}_R U_q(\mathfrak{g}) \rightarrow \text{End}(V^{\otimes n})$  which comes from the action of  ${}_R U_q(\mathfrak{g})$  on  $V^{\otimes n}$ . Then some relations between  ${}_R \mathcal{S}_{n,r}^0$  and  ${}_R \mathcal{S}_{n,r}$  are studied in [SawS] and [Saw]. In particular,  ${}_R \mathcal{S}_{n,r}^0$  is realized as a subquotient algebra of  ${}_R \mathcal{S}_{n,r}$ . Then, some decomposition numbers of  ${}_R \mathcal{S}_{n,r}$  coincide with the decomposition numbers of  ${}_R \mathcal{S}_{n,r}^0$  (which are also decomposition number for  ${}_R U_q(\mathfrak{g})$ ) when  $R$  is a field. In [SW], we obtained a certain generalization of these results (see also Remark 5.7). Motivated by this generalization together with the Schur-Weyl duality between  ${}_R U_q(\mathfrak{g})$  and  ${}_R \mathcal{H}_{n,r}^0$ , the author gave a presentation of  $\mathcal{S}_{n,r}$  (also  ${}_{\mathcal{A}} \mathcal{S}_{n,r}$ ) by generators and fundamental relations in [W]. By using this presentation, we can define a (not surjective) homomorphism  $\Phi_{\mathfrak{g}} : U_q(\mathfrak{g}) \rightarrow \mathcal{S}_{n,r}$ . We also have  $\Phi_{\mathfrak{g}}|_{{}_{\mathcal{A}} U_q(\mathfrak{g})} : {}_{\mathcal{A}} U_q(\mathfrak{g}) \rightarrow {}_{\mathcal{A}} \mathcal{S}_{n,r}$  by restriction. Thus we can specialize it to any commutative ring  $R$  and parameters  $q, Q_1, \dots, Q_r \in R$ . In this paper, we study  ${}_R \mathcal{S}_{n,r}$ -modules by restricting the action to  ${}_R U_q(\mathfrak{g})$  when  $R$  is a field.

First, we consider over  $\mathcal{K}$ . In this case,  $\mathcal{S}_{n,r}$  is semi-simple, and finite dimensional  $U_q(\mathfrak{g})$ -module is also semi-simple. Put  $\Lambda_{n,r}^+ = \{\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \mid \lambda^{(k)} : \text{partition, } \sum_{k=1}^r |\lambda^{(k)}| = n\}$ , the set of  $r$ -partitions of size  $n$ . Let  $W(\lambda)$  be the Weyl module of  $\mathcal{S}_{n,r}$  corresponding to  $\lambda \in \Lambda_{n,r}^+$ . It is well known that  $\{W(\lambda) \mid \lambda \in \Lambda_{n,r}^+\}$  gives a complete set of non-isomorphic simple  $\mathcal{S}_{n,r}$ -modules. By investigating the appearing weights, we see that  $\{W(\lambda^{(1)}) \boxtimes \dots \boxtimes W(\lambda^{(r)}) \mid \lambda \in \Lambda_{n,r}^+\}$  gives a complete set of non-isomorphic simple  $U_q(\mathfrak{g})$ -modules which appear as  $U_q(\mathfrak{g})$ -submodules of  $\mathcal{S}_{n,r}$ -modules through the homomorphism  $\Phi_{\mathfrak{g}}$ , where  $W(\lambda^{(k)})$  is the Weyl module of  $U_q(\mathfrak{gl}_{m_k})$  with the highest weight  $\lambda^{(k)}$ . Then we can consider the irreducible decomposition of the Weyl module  $W(\lambda)$  of  $\mathcal{S}_{n,r}$  as  $U_q(\mathfrak{g})$ -modules through the homomorphism  $\Phi_{\mathfrak{g}}$  as follows:

$$(0.1) \quad W(\lambda) \cong \bigoplus_{\mu \in \Lambda_{n,r}^+} \left( W(\mu^{(1)}) \boxtimes \dots \boxtimes W(\mu^{(r)}) \right)^{\oplus \beta_{\lambda\mu}} \text{ as } U_q(\mathfrak{g})\text{-modules.}$$

In order to compute the multiplicity  $\beta_{\lambda\mu}$  in this decomposition, we describe the  $U_q(\mathfrak{g})$ -crystal structure on  $W(\lambda)$  by using a generalization of “admissible reading” for  $U_q(\mathfrak{gl}_m)$ -crystal given in [KN] (Theorem 2.17). As a consequence, we can compute the multiplicity  $\beta_{\lambda\mu}$  by the combinatorial way which can be regarded as a generalization of the Littlewood-Richardson rule (Corollary 3.7. See also Remark 3.8).

Thanks to the decomposition (0.1), we obtain the character formula of  $W(\lambda)$  by using Kostka numbers and multiplicities  $\beta_{\lambda\mu}$  ( $\lambda, \mu \in \Lambda_{n,r}^+$ ) (Note that the weight space as the  $\mathcal{S}_{n,r}$ -module coincides with the weight space as the  $U_q(\mathfrak{g})$ -module from the homomorphism  $\Phi_{\mathfrak{g}}$ ). We also describe the character of  $W(\lambda)$  as a linear combination of the products of Schur polynomials with coefficients  $\beta_{\lambda\mu}$  ( $\lambda, \mu \in \Lambda_{n,r}^+$ ). Moreover, we see that the set of characters of the Weyl modules for all  $r$ -partitions gives a new basis of the ring of symmetric polynomials (Theorem 4.3). Then we also study on some properties for such symmetric functions.

As an application of the decomposition (0.1), we have a certain factorization of decomposition matrix of  ${}_R\mathcal{S}_{n,r}$  when  $R$  is a field (Theorem 5.5), and we give an alternative proof of the product formula for decomposition numbers of  ${}_R\mathcal{S}_{n,r}$  given by [Saw] (Corollary 5.6, See also Remark 5.7.). For special parameters ( $Q_1 = \cdots = Q_r = 0$  or  $q = 1$ ,  $Q_1 = \cdots = Q_r$ ), we can compute the decomposition matrix of  ${}_R\mathcal{S}_{n,r}$  from the factorization of decomposition matrix (Corollary 5.8).

Finally, we realize the Ariki-Koike algebra  ${}_R\mathcal{H}_{n,r}$  as a subalgebra of  ${}_R\mathcal{S}_{n,r}$  by using the generators of  ${}_R\mathcal{S}_{n,r}$  (Proposition 6.3), and we give an alternative proof for the classification of simple  $\mathcal{H}_{n,r}$ -modules for special parameters ( $Q_1 = \cdots = Q_r = 0$  or  $q = 1$ ,  $Q_1 = \cdots = Q_r$ ) which has already obtained by [AM] and [M1] (Corollary 6.5).

**Acknowledgments :** The author is grateful to Professors S. Ariki, H. Miyachi and T. Shoji for many valuable discussions and comments.

## § 1. REVIEW ON CYCLOTOMIC $q$ -SCHUR ALGEBRAS

In this section, we recall the definition of the cyclotomic  $q$ -Schur algebra  $\mathcal{S}_{n,r}$  introduced by [DJM], and we review presentations of  $\mathcal{S}_{n,r}$  by generators and fundamental relations given by [W].

**1.1.** Let  $R$  be a commutative ring, and we take parameters  $q, Q_1, \dots, Q_r \in R$  such that  $q$  is invertible in  $R$ . The Ariki-Koike algebra  ${}_R\mathcal{H}_{n,r}$  associated to the complex reflection group  $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$  is the associative algebra with 1 over  $R$  generated by  $T_0, T_1, \dots, T_{n-1}$  with the following defining relations:

$$\begin{aligned} (T_0 - Q_1)(T_0 - Q_2) \cdots (T_0 - Q_r) &= 0, \\ (T_i - q)(T_i + q^{-1}) &= 0 & (1 \leq i \leq n-1), \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & (1 \leq i \leq n-2), \\ T_i T_j &= T_j T_i & (|i - j| \geq 2). \end{aligned}$$

The subalgebra of  ${}_R\mathcal{H}_{n,r}$  generated by  $T_1, \dots, T_{n-1}$  is isomorphic to the Iwahori-Hecke algebra  ${}_R\mathcal{H}_n$  of the symmetric group  $\mathfrak{S}_n$  of degree  $n$ . For  $w \in \mathfrak{S}_n$ , we denote by  $\ell(w)$  the length of  $w$ , and denote by  $T_w$  the standard basis of  ${}_R\mathcal{H}_n$  corresponding to  $w$ .

**1.2.** Let  $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{>0}^r$  be an  $r$ -tuple of positive integers. Put

$$\Lambda_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \dots, \mu^{(r)}) \left| \begin{array}{l} \mu^{(k)} = (\mu_1^{(k)}, \dots, \mu_{m_k}^{(k)}) \in \mathbb{Z}_{\geq 0}^{m_k} \\ \sum_{k=1}^r \sum_{i=1}^{m_k} \mu_i^{(k)} = n \end{array} \right. \right\}.$$

We denote by  $|\mu^{(k)}| = \sum_{i=1}^{m_k} \mu_i^{(k)}$  (resp.  $|\mu| = \sum_{k=1}^r |\mu^{(k)}|$ ) the size of  $\mu^{(k)}$  (resp. the size of  $\mu$ ), and call an element of  $\Lambda_{n,r}(\mathbf{m})$  an  $r$ -composition of size  $n$ . We define the map  $\zeta : \Lambda_{n,r}(\mathbf{m}) \rightarrow \mathbb{Z}_{\geq 0}^r$  by  $\zeta(\mu) = (|\mu^{(1)}|, |\mu^{(2)}|, \dots, |\mu^{(r)}|)$  for  $\mu \in \Lambda_{n,r}(\mathbf{m})$ . We also define the partial order " $\succeq$ " on  $\mathbb{Z}_{\geq 0}^r$  by  $(a_1, \dots, a_r) \succ (a'_1, \dots, a'_r)$  if  $\sum_{j=1}^k a_j \geq \sum_{j=1}^k a'_j$  for all  $k = 1, \dots, r$ .

$\sum_{j=1}^k a'_j$  for any  $k = 1, \dots, r$ . Put  $\Lambda_{n,r}^+(\mathbf{m}) = \{\lambda \in \Lambda_{n,r}(\mathbf{m}) \mid \lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \dots \geq \lambda_{m_k}^{(k)} \text{ for any } k = 1, \dots, r\}$ . We also denote by  $\Lambda_{n,r}^+$  the set of  $r$ -partitions of size  $n$ . Then we have  $\Lambda_{n,r}^+(\mathbf{m}) = \Lambda_{n,r}^+$  when  $m_k \geq n$  for any  $k = 1, \dots, r$ .

**1.3.** For  $i = 1, \dots, n$ , put  $L_1 = T_0$  and  $L_i = T_{i-1}L_{i-1}T_{i-1}$ . For  $\mu \in \Lambda_{n,r}(\mathbf{m})$ , put

$$m_\mu = \left( \sum_{w \in \mathfrak{S}_\mu} q^{\ell(w)} T_w \right) \left( \prod_{k=1}^r \prod_{i=1}^{a_k} (L_i - Q_k) \right), \quad M^\mu = m_\mu \cdot {}_R\mathcal{H}_{n,r},$$

where  $\mathfrak{S}_\mu$  is the Young subgroup of  $\mathfrak{S}_n$  with respect to  $\mu$ , and  $a_k = \sum_{j=1}^{k-1} |\mu^{(j)}|$  with  $a_1 = 0$ . The cyclotomic  $q$ -Schur algebra  ${}_R\mathcal{S}_{n,r}$  associated to  ${}_R\mathcal{H}_{n,r}$  is defined by

$${}_R\mathcal{S}_{n,r} = {}_R\mathcal{S}_{n,r}(\Lambda_{n,r}(\mathbf{m})) = \text{End}_{{}_R\mathcal{H}_{n,r}} \left( \bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} M^\mu \right).$$

Put  $\Gamma(\mathbf{m}) = \{(i, k) \mid 1 \leq i \leq m_k, 1 \leq k \leq r\}$ . For  $\mu \in \Lambda_{n,r}(\mathbf{m})$  and  $(i, k) \in \Gamma(\mathbf{m})$ , we define  $\sigma_{(i,k)}^\mu \in {}_R\mathcal{S}_{n,r}$  by

$$\sigma_{(i,k)}^\mu(m_\nu \cdot h) = \delta_{\mu,\nu} (m_\mu(L_{N+1} + L_{N+2} + \dots + L_{N+\mu_i^{(k)}})) \cdot h \quad (\nu \in \Lambda_{n,r}(\mathbf{m}), h \in {}_R\mathcal{H}_{n,r}),$$

where  $N = \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{j=1}^{i-1} \mu_j^{(k)}$ , and we set  $\sigma_{(i,k)}^\mu = 0$  if  $\mu_i^{(k)} = 0$ . For  $(i, k) \in \Gamma(\mathbf{m})$ , put  $\sigma_{(i,k)} = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \sigma_{(i,k)}^\mu$ , then  $\sigma_{(i,k)}$  is a Jucys-Murphy element of  ${}_R\mathcal{S}_{n,r}$  (See [M2] for properties of Jucys-Murphy elements).

**1.4.** Let  $\mathcal{A} = \mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_r]$ , where  $q, Q_1, \dots, Q_r$  are indeterminate over  $\mathbb{Z}$ , and  $\mathcal{K} = \mathbb{Q}(q, Q_1, \dots, Q_r)$  be the quotient field of  $\mathcal{A}$ . In order to describe presentations of  ${}_K\mathcal{S}_{n,r}$  (resp.  ${}_A\mathcal{S}_{n,r}$ ), we prepare some notation.

Put  $m = \sum_{k=1}^r m_k$ . Let  $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$  be the weight lattice of  $\mathfrak{gl}_m$ , and  $P^\vee = \bigoplus_{i=1}^m \mathbb{Z}h_i$  be the dual weight lattice with the natural pairing  $\langle \cdot, \cdot \rangle : P \times P^\vee \rightarrow \mathbb{Z}$  such that  $\langle \varepsilon_i, h_j \rangle = \delta_{ij}$ . Set  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for  $i = 1, \dots, m-1$ , then  $\Pi = \{\alpha_i \mid 1 \leq i \leq m-1\}$  is the set of simple roots, and  $Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i$  is the root lattice of  $\mathfrak{gl}_m$ . Put  $Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\geq 0} \alpha_i$ . We define a partial order “ $\geq$ ” on  $P$ , so called dominance order, by  $\lambda \geq \mu$  if  $\lambda - \mu \in Q^+$ .

We identify the set  $\Gamma(\mathbf{m})$  with the set  $\{1, \dots, m\}$  by the bijection  $\gamma : \Gamma(\mathbf{m}) \rightarrow \{1, \dots, m\}$  given by  $\gamma((i, k)) = \sum_{j=1}^{k-1} m_j + i$ . Put  $\Gamma'(\mathbf{m}) = \Gamma \setminus \{(m_r, r)\}$ . Under this identification, we have  $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i = \bigoplus_{(i,k) \in \Gamma(\mathbf{m})} \mathbb{Z}\varepsilon_{(i,k)}$  and  $Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i = \bigoplus_{(i,k) \in \Gamma'(\mathbf{m})} \mathbb{Z}\alpha_{(i,k)}$ . Then we regard  $\Lambda_{n,r}(\mathbf{m})$  as a subset of  $P$  by the injective map  $\lambda \mapsto \sum_{(i,k) \in \Gamma(\mathbf{m})} \lambda_i^{(k)} \varepsilon_{(i,k)}$ . For convenience, we consider  $(m_k + 1, k) = (1, k + 1)$  for  $(m_k, k) \in \Gamma'(\mathbf{m})$  (resp.  $(1 - 1, k) = (m_{k-1}, k - 1)$  for  $(1, k) \in \Gamma(\mathbf{m}) \setminus \{(1, 1)\}$ ).

Now we have the following two presentations of cyclotomic  $q$ -Schur algebras.

**Theorem 1.5** ([W, Theorem 7.16]). *Assume that  $m_k \geq n$  for any  $k = 1, \dots, r$ , we have the following presentations of  ${}_K\mathcal{S}_{n,r}$  and  ${}_A\mathcal{S}_{n,r}$ .*

(i)  ${}_{\mathcal{K}}\mathcal{S}_{n,r}$  is isomorphic to the algebra over  $\mathcal{K}$  defined by the generators  $e_{(i,k)}, f_{(i,k)}$   $((i,k) \in \Gamma'(\mathbf{m}))$  and  $K_{(i,k)}^{\pm}$   $((i,k) \in \Gamma(\mathbf{m}))$  with the following defining relations :

$$(1.5.1) \quad K_{(i,k)}K_{(j,l)} = K_{(j,l)}K_{(i,k)}, \quad K_{(i,k)}K_{(i,k)}^{-} = K_{(i,k)}^{-}K_{(i,k)} = 1$$

$$(1.5.2) \quad K_{(i,k)}e_{(j,l)}K_{(i,k)}^{-} = q^{\langle \alpha_{(j,l)}, h_{(i,k)} \rangle} e_{(j,l)},$$

$$(1.5.3) \quad K_{(i,k)}f_{(j,l)}K_{(i,k)}^{-} = q^{-\langle \alpha_{(j,l)}, h_{(i,k)} \rangle} f_{(j,l)},$$

$$(1.5.4) \quad e_{(i,k)}f_{(j,l)} - f_{(j,l)}e_{(i,k)} = \delta_{(i,k),(j,l)}\eta_{(i,k)},$$

$$\text{where } \eta_{(i,k)} = \begin{cases} \frac{K_{(i,k)}K_{(i+1,k)}^{-} - K_{(i,k)}^{-}K_{(i+1,k)}}{q - q^{-1}} & \text{if } i \neq m_k, \\ -Q_{k+1} \frac{K_{(m_k,k)}K_{(1,k+1)}^{-} - K_{(m_k,k)}^{-}K_{(1,k+1)}}{q - q^{-1}} \\ \quad + K_{(m_k,k)}K_{(1,k+1)}^{-}(q^{-1}g_{(m_k,k)}(f,e) - qg_{(1,k+1)}(f,e)) & \text{if } i = m_k, \end{cases}$$

$$(1.5.5) \quad e_{(i\pm 1,k)}e_{(i,k)}^2 - (q + q^{-1})e_{(i,k)}e_{(i\pm 1,k)}e_{(i,k)} + e_{(i,k)}^2e_{(i\pm 1,k)} = 0,$$

$$e_{(i,k)}e_{(j,l)} = e_{(j,l)}e_{(i,k)} \quad (|\gamma((i,k)) - \gamma((j,l))| \geq 2),$$

$$(1.5.6) \quad f_{(i\pm 1,k)}f_{(i,k)}^2 - (q + q^{-1})f_{(i,k)}f_{(i\pm 1,k)}f_{(i,k)} + f_{(i,k)}^2f_{(i\pm 1,k)} = 0,$$

$$f_{(i,k)}f_{(j,l)} = f_{(j,l)}f_{(i,k)} \quad (|\gamma((i,k)) - \gamma((j,l))| \geq 2),$$

$$(1.5.7) \quad \prod_{(i,k) \in \Gamma(\mathbf{m})} K_{(i,k)} = q^n,$$

$$(1.5.8) \quad (K_{(i,k)} - 1)(K_{(i,k)} - q)(K_{(i,k)} - q^2) \cdots (K_{(i,k)} - q^n) = 0.$$

The elements  $g_{(m_k,k)}(f,e)$ ,  $g_{(1,k+1)}(f,e)$  in (1.5.4) coincide with the Jucys-Murphy elements  $\sigma_{(m_k,k)}$ ,  $\sigma_{(1,k+1)}$  respectively, which are described by generators  $e_{(i,k)}, f_{(i,k)}$   $((i,k) \in \Gamma'(\mathbf{m}))$  and  $K_{(i,k)}^{\pm}$   $((i,k) \in \Gamma(\mathbf{m}))$  (see [W, 7.11]).

Moreover,  ${}_{\mathcal{A}}\mathcal{S}_{n,r}$  is isomorphic to the  $\mathcal{A}$ -subalgebra of  ${}_{\mathcal{K}}\mathcal{S}_{n,r}$  generated by  $e_{(i,k)}^l/[l]!$ ,  $f_{(i,k)}^l/[l]!$   $((i,k) \in \Gamma'(\mathbf{m}), l \geq 1)$ ,  $K_{(i,k)}^{\pm}$ ,  $\left[ \begin{matrix} K_{(i,k)} \\ t \end{matrix} ; 0 \right] = \prod_{s=1}^t \frac{K_{(i,k)}q^{-s+1} - K_{(i,k)}^{-1}q^{s-1}}{q^s - q^{-s}}$   $((i,k) \in \Gamma(\mathbf{m}), t \geq 1)$ , where  $[l] = \frac{q^l - q^{-l}}{q - q^{-1}}$  and  $[l]! = [l][l-1] \cdots [1]$ .

(ii)  ${}_{\mathcal{K}}\mathcal{S}_{n,r}$  is isomorphic to the algebra over  $\mathcal{K}$  defined by the generators  $E_{(i,k)}$ ,  $F_{(i,k)}$   $((i,k) \in \Gamma'(\mathbf{m}))$ ,  $1_{\lambda}$   $(\lambda \in \Lambda_{n,r}(\mathbf{m}))$  with the following defining relations:

$$(1.5.9) \quad 1_{\lambda}1_{\mu} = \delta_{\lambda,\mu}1_{\lambda}, \quad \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_{\lambda} = 1,$$

$$(1.5.10) \quad E_{(i,k)}1_{\lambda} = \begin{cases} 1_{\lambda + \alpha_{(i,k)}}E_{(i,k)} & \text{if } \lambda + \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.5.11) \quad F_{(i,k)}1_{\lambda} = \begin{cases} 1_{\lambda - \alpha_{(i,k)}}F_{(i,k)} & \text{if } \lambda - \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.5.12) \quad 1_\lambda E_{(i,k)} = \begin{cases} E_{(i,k)} 1_{\lambda - \alpha_{(i,k)}} & \text{if } \lambda - \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.5.13) \quad 1_\lambda F_{(i,k)} = \begin{cases} F_{(i,k)} 1_{\lambda + \alpha_{(i,k)}} & \text{if } \lambda + \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.5.14) \quad E_{(i,k)} F_{(j,l)} - F_{(j,l)} E_{(i,k)} = \delta_{(i,k),(j,l)} \sum_{\lambda \in \Lambda_{n,r}} \eta_{(i,k)}^\lambda,$$

$$\text{where } \eta_{(i,k)}^\lambda = \begin{cases} [\lambda_i^{(k)} - \lambda_{i+1}^{(k)}] 1_\lambda & \text{if } i \neq m_k, \\ \left( -Q_{k+1} [\lambda_{m_k}^{(k)} - \lambda_1^{(k+1)}] \right. \\ \quad \left. + q^{\lambda_{m_k}^{(k)} - \lambda_1^{(k+1)}} (q^{-1} g_{(m_k,k)}^\lambda(F, E) - q g_{(1,k+1)}^\lambda(F, E)) \right) 1_\lambda & \text{if } i = m_k, \end{cases}$$

$$(1.5.15) \quad E_{(i\pm 1,k)} (E_{(i,k)})^2 - (q + q^{-1}) E_{(i,k)} E_{(i\pm 1,k)} E_{(i,k)} + (E_{(i,k)})^2 E_{(i\pm 1,k)} = 0,$$

$$E_{(i,k)} E_{(j,l)} = E_{(j,l)} E_{(i,k)} \quad (|\gamma((i,k)) - \gamma((j,l))| \geq 2),$$

$$(1.5.16) \quad F_{(i\pm 1,k)} (F_{(i,k)})^2 - (q + q^{-1}) F_{(i,k)} F_{(i\pm 1,k)} F_{(i,k)} + (F_{(i,k)})^2 F_{(i\pm 1,k)} = 0,$$

$$F_{(i,k)} F_{(j,l)} = F_{(j,l)} F_{(i,k)} \quad (|\gamma((i,k)) - \gamma((j,l))| \geq 2),$$

The elements  $g_{(m_k,k)}^\lambda(F, E)$ ,  $g_{(1,k+1)}^\lambda(F, E)$  in (1.5.14) coincide with  $\sigma_{(m_k,k)}^\lambda$ ,  $\sigma_{(1,k+1)}^\lambda$  respectively, which are described by generators  $E_{(i,k)}, F_{(i,k)}$   $((i,k) \in \Gamma'(\mathbf{m}))$  (see [W, 7.1-7.4]).

Moreover,  $\mathcal{A}\mathcal{S}_{n,r}$  is isomorphic to the  $\mathcal{A}$ -subalgebra of  ${}_{\mathcal{K}}\mathcal{S}_{n,r}$  generated by  $E_{(i,k)}^l/[l]!$ ,  $F_{(i,k)}^l/[l]!$   $((i,k) \in \Gamma'(\mathbf{m}), l \geq 1)$ ,  $1_\lambda$   $(\lambda \in \Lambda_{n,r}(\mathbf{m}))$ .

**Remark 1.6.** In [W], we treated only the case where  $m_k = n$  for any  $k = 1, \dots, r$ . We obtain Theorem 1.5 for the general case in the same way under the condition  $m_k \geq n$  for any  $k = 1, \dots, r$ . However, in the case where  $m_k < n$  for some  $k$ , we do not have the presentation of  $\mathcal{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$  as in the above theorem. In such a case, we have the following realization of  $\mathcal{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$ . First, we take  $\tilde{\mathbf{m}} = (\tilde{m}_1, \dots, \tilde{m}_r) \in \mathbb{Z}_{>0}^r$  such that  $\tilde{m}_k \geq n$  and  $\tilde{m}_k \geq m_k$  for any  $k = 1, \dots, r$ . Then, we can regard  $\Lambda_{n,r}(\mathbf{m})$  as a subset of  $\Lambda_{n,r}(\tilde{\mathbf{m}})$  in the natural way. We have the presentation of  $\mathcal{S}_{n,r}(\Lambda_{n,r}(\tilde{\mathbf{m}}))$  by the theorem, and we have  $\mathcal{S}_{n,r}(\Lambda_{n,r}(\mathbf{m})) = 1_{\mathbf{m}} \mathcal{S}_{n,r}(\Lambda_{n,r}(\tilde{\mathbf{m}})) 1_{\mathbf{m}}$ , where  $1_{\mathbf{m}} = \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda \in \mathcal{S}_{n,r}(\Lambda_{n,r}(\tilde{\mathbf{m}}))$ .

**1.7. Weyl modules** (see [W] and [DJM] for more details). Let  $\mathcal{A}\mathcal{S}_{n,r}^+$  (resp.  $\mathcal{A}\mathcal{S}_{n,r}^-$ ) be the subalgebra of  $\mathcal{A}\mathcal{S}_{n,r}$  generated by  $E_{(i,k)}^l/[l]!$  (resp.  $F_{(i,k)}^l/[l]!$ ) for  $(i,k) \in \Gamma'(\mathbf{m})$  and  $l \geq 1$ . Let  $\mathcal{A}\mathcal{S}_{n,r}^0$  be the subalgebra of  $\mathcal{A}\mathcal{S}_{n,r}$  generated by  $1_\lambda$  for  $\lambda \in \Lambda_{n,r}(\mathbf{m})$ . Then  $\mathcal{A}\mathcal{S}_{n,r}$  has the triangular decomposition  $\mathcal{A}\mathcal{S}_{n,r} = \mathcal{A}\mathcal{S}_{n,r}^- \mathcal{A}\mathcal{S}_{n,r}^0 \mathcal{A}\mathcal{S}_{n,r}^+$ . We denote by  $\mathcal{A}\mathcal{S}_{n,r}^{\geq 0}$  the subalgebra of  $\mathcal{A}\mathcal{S}_{n,r}$  generated by  $\mathcal{A}\mathcal{S}_{n,r}^+$  and  $\mathcal{A}\mathcal{S}_{n,r}^0$ .

Let  $R$  be an arbitrary commutative ring, and we take parameters  $\tilde{q}, \tilde{Q}_1, \dots, \tilde{Q}_r \in R$  such that  $\tilde{q}$  is invertible in  $R$ . We consider the specialized cyclotomic  $q$ -Schur algebra  ${}_R\mathcal{S}_{n,r} = R \otimes_{\mathcal{A}} \mathcal{A}\mathcal{S}_{n,r}$  through the ring homomorphism  $\mathcal{A} \rightarrow R$ ,  $q \mapsto \tilde{q}$ ,  $Q_k \mapsto \tilde{Q}_k$  ( $1 \leq k \leq r$ ). Then  ${}_R\mathcal{S}_{n,r}$  also has the triangular decomposition  ${}_R\mathcal{S}_{n,r} = {}_R\mathcal{S}_{n,r}^- {}_R\mathcal{S}_{n,r}^0 {}_R\mathcal{S}_{n,r}^+$  which comes from the triangular decomposition of  $\mathcal{A}\mathcal{S}_{n,r}$ .



For  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ , we define the 1-dimensional  ${}_R\mathcal{S}_{n,r}^{\geq 0}$ -module  $\theta_\lambda = Rv_\lambda$  by  $E_{(i,k)} \cdot v_\lambda = 0$  ( $(i,k) \in \Gamma'(\mathbf{m})$ ) and  $1_\mu \cdot v_\lambda = \delta_{\lambda,\mu} v_\lambda$  ( $\mu \in \Lambda_{n,r}(\mathbf{m})$ ). Then the Weyl module  ${}_RW(\lambda)$  of  ${}_R\mathcal{S}_{n,r}$  is defined as the induced module  ${}_R\mathcal{S}_{n,r} \otimes_{{}_R\mathcal{S}_{n,r}^{\geq 0}} \theta_\lambda$  of  $\theta_\lambda$  for  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ .

When  $R$  is a field, it is known that  ${}_RW(\lambda)$  has the unique simple top  ${}_RL(\lambda)$ , and that  $\{ {}_RL(\lambda) \mid \lambda \in \Lambda_{n,r}^+(\mathbf{m}) \}$  gives a complete set of non-isomorphic (left) simple  ${}_R\mathcal{S}_{n,r}$ -modules. Moreover, it is known that  ${}_K\mathcal{S}_{n,r}$  is semi-simple, and that  $\{ {}_KW(\lambda) \mid \lambda \in \Lambda_{n,r}^+(\mathbf{m}) \}$  gives a complete set of non-isomorphic (left) simple  ${}_K\mathcal{S}_{n,r}$ -modules.

**1.8.** By (1.5.9), the identity element 1 of  ${}_R\mathcal{S}_{n,r}$  decomposes to a sum of pairwise orthogonal idempotents indexed by  $\Lambda_{n,r}(\mathbf{m})$ , namely we have  $1 = \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda$ . Thanks to this decomposition, for  ${}_R\mathcal{S}_{n,r}$ -module  $M$ , we have the decomposition  $M = \bigoplus_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda M$  as  $R$ -modules. By the isomorphism between the first presentation and the second presentation of  $\mathcal{S}_{n,r}$  in Theorem 1.5 (see [W, Proposition 7.12] for this isomorphism), we see that  $K_{(i,k)}$  acts on  $1_\lambda M$  by multiplying the scalar  $q^{\lambda_i^{(k)}}$ , namely we have  $1_\lambda M = \{ m \in M \mid K_{(i,k)} \cdot m = q^{\lambda_i^{(k)}} m \text{ for } (i,k) \in \Gamma(\mathbf{m}) \}$ . We call  $1_\lambda M$  the weight space of weight  $\lambda$  (or  $\lambda$ -weight space simply), and denote by  $M_\lambda$ .

## § 2. $U_q(\mathfrak{g})$ -CRYSTAL STRUCTURE ON WEYL MODULE $W(\lambda)$ OF $\mathcal{S}_{n,r}$

In the section 2 – section 4, we consider only the cyclotomic  $q$ -Schur algebra  ${}_K\mathcal{S}_{n,r}$  over  $\mathcal{K}$ . Hence, we omit the subscript  $\mathcal{K}$ . Moreover, we assume that  $m_k \geq n$  for any  $k = 1, \dots, r$  in this section.

**2.1.** Let  $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \dots \oplus \mathfrak{gl}_{m_r}$  be the Levi subalgebra of  $\mathfrak{gl}_m$ , and  $U_q(\mathfrak{g}) \cong U_q(\mathfrak{gl}_{m_1}) \otimes \dots \otimes U_q(\mathfrak{gl}_{m_r})$  be the quantum group over  $\mathcal{K}$  corresponding to  $\mathfrak{g}$ . Put  $\Gamma'_\mathfrak{g}(\mathbf{m}) = \Gamma(\mathbf{m}) \setminus \{ (m_k, k) \mid 1 \leq k \leq r \}$ . Let  $e_{(i,k)}, f_{(i,k)}$  ( $(i,k) \in \Gamma'_\mathfrak{g}(\mathbf{m})$ ),  $K_{(i,k)}^\pm$  ( $(i,k) \in \Gamma(\mathbf{m})$ ) be the generators of  $U_q(\mathfrak{g})$ , where  $e_{(i,k)}, f_{(i,k)}, K_{(j,k)}^\pm$  ( $1 \leq i \leq m_k - 1, 1 \leq j \leq m_k$ ) is the usual Chevalley generators of  $U_q(\mathfrak{gl}_{m_k})$ .

By the presentation of  $\mathcal{S}_{n,r}$  (Theorem 1.5), we can define the algebra homomorphism  $\Phi_\mathfrak{g} : U_q(\mathfrak{g}) \rightarrow \mathcal{S}_{n,r}$  sending generators of  $U_q(\mathfrak{g})$  to the corresponding generators of  $\mathcal{S}_{n,r}$  denoted by the same symbol. Note that  $\Phi_\mathfrak{g}$  is not surjective without the case where  $r = 1$ . We have the following lemma which describes the image of  $\Phi_\mathfrak{g}$ .

**Lemma 2.2.**

$$(i) \quad \Phi_\mathfrak{g}(U_q(\mathfrak{g})) \cong \bigoplus_{\substack{\eta = (n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \mathcal{S}_{n_1,1}^\eta(\Lambda_{n_1,1}(m_1)) \otimes \dots \otimes \mathcal{S}_{n_r,1}^\eta(\Lambda_{n_r,1}(m_r)),$$

where  $\mathcal{S}_{n_k,1}^\eta(\Lambda_{n_k,1}(m_k))$  is the  $q$ -Schur algebra associated to the symmetric group  $\mathfrak{S}_{n_k}$  of degree  $n_k$ .

(ii) Let  ${}_A U_q(\mathfrak{g})$  be the  $\mathcal{A}$ -form of  $U_q(\mathfrak{g})$  by taking the divided powers. Then we have

$$\Phi_\mathfrak{g}({}_A U_q(\mathfrak{g})) \cong \bigoplus_{\substack{\eta = (n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} {}_A \mathcal{S}_{n_1,1}^\eta(\Lambda_{n_1,1}(m_1)) \otimes \dots \otimes {}_A \mathcal{S}_{n_r,1}^\eta(\Lambda_{n_r,1}(m_r))$$

*Proof.* Let  $e_i^{\eta_k}, f_i^{\eta_k}$  ( $1 \leq i \leq m_k - 1$ ),  $K_i^{\eta_k \pm}$  ( $1 \leq i \leq m_k$ ) be the generators of  $\mathcal{S}_{n_k,1}^\eta(\Lambda_{n_k,1}(m_k))$  in Theorem 1.5 (i). Then, we define the homomorphism of algebras

$$\varphi : U_q(\mathfrak{g}) \rightarrow \bigoplus_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \mathcal{S}_{n_1,1}^\eta(\Lambda_{n_1,1}(m_1)) \otimes \dots \otimes \mathcal{S}_{n_r,1}^\eta(\Lambda_{n_r,1}(m_r))$$

by

$$\begin{aligned} \varphi(e_{(i,k)}) &= \sum_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \underbrace{1 \otimes \dots \otimes 1}_{k-1} \otimes e_i^{\eta_k} \otimes 1 \otimes \dots \otimes 1, \\ \varphi(f_{(i,k)}) &= \sum_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \underbrace{1 \otimes \dots \otimes 1}_{k-1} \otimes f_i^{\eta_k} \otimes 1 \otimes \dots \otimes 1, \\ \varphi(K_{(i,k)}^\pm) &= \sum_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \underbrace{1 \otimes \dots \otimes 1}_{k-1} \otimes K_i^{\eta_k \pm} \otimes 1 \otimes \dots \otimes 1 \end{aligned}$$

for generators  $e_{(i,k)}, f_{(i,k)}$  ( $(i,k) \in \Gamma'_g(\mathbf{m})$ ),  $K_{(i,k)}^\pm$  ( $(i,k) \in \Gamma(\mathbf{m})$ ) of  $U_q(\mathfrak{g})$ . (We can easily check that  $\varphi$  is well-defined by Theorem 1.5 (i).)

We also define the homomorphism of algebras

$$\psi : \bigoplus_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \mathcal{S}_{n_1,1}^\eta(\Lambda_{n_1,1}(m_1)) \otimes \dots \otimes \mathcal{S}_{n_r,1}^\eta(\Lambda_{n_r,1}(m_r)) \rightarrow \Phi_g(U_q(\mathfrak{g}))$$

by

$$\begin{aligned} \psi(\underbrace{1 \otimes \dots \otimes 1}_{k-1} \otimes e_i^{\eta_k} \otimes 1 \otimes \dots \otimes 1) &= \left( \sum_{\substack{\mu \in \Lambda_{n,r}(\mathbf{m}) \\ \zeta(\mu) = \eta}} 1_\mu \right) \cdot \Phi(e_{(i,k)}) \cdot \left( \sum_{\substack{\mu \in \Lambda_{n,r}(\mathbf{m}) \\ \zeta(\mu) = \eta}} 1_\mu \right), \\ \psi(\underbrace{1 \otimes \dots \otimes 1}_{k-1} \otimes f_i^{\eta_k} \otimes 1 \otimes \dots \otimes 1) &= \left( \sum_{\substack{\mu \in \Lambda_{n,r}(\mathbf{m}) \\ \zeta(\mu) = \eta}} 1_\mu \right) \cdot \Phi(f_{(i,k)}) \cdot \left( \sum_{\substack{\mu \in \Lambda_{n,r}(\mathbf{m}) \\ \zeta(\mu) = \eta}} 1_\mu \right), \\ \psi(\underbrace{1 \otimes \dots \otimes 1}_{k-1} \otimes K_i^{\eta_k \pm} \otimes 1 \otimes \dots \otimes 1) &= \left( \sum_{\substack{\mu \in \Lambda_{n,r}(\mathbf{m}) \\ \zeta(\mu) = \eta}} 1_\mu \right) \cdot \Phi(K_{(i,k)}) \cdot \left( \sum_{\substack{\mu \in \Lambda_{n,r}(\mathbf{m}) \\ \zeta(\mu) = \eta}} 1_\mu \right) \end{aligned}$$

for each generators of  $\bigoplus_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \mathcal{S}_{n_1,1}^\eta(\Lambda_{n_1,1}(m_1)) \otimes \dots \otimes \mathcal{S}_{n_r,1}^\eta(\Lambda_{n_r,1}(m_r))$ . (We can check the well-definedness by direct calculations.) It is clear that  $\psi \circ \varphi = \Phi_g$ . Thus,  $\psi$  is surjective. Moreover, by comparing the simple modules appearing in  $\bigoplus_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \mathcal{S}_{n_1,1}^\eta(\Lambda_{n_1,1}(m_1)) \otimes \dots \otimes \mathcal{S}_{n_r,1}^\eta(\Lambda_{n_r,1}(m_r))$  and in  $\Phi_g(U_q(\mathfrak{g}))$  as  $U_q(\mathfrak{g})$ -modules through  $\varphi$  and  $\Phi_g$  respectively, we see that  $\psi$  is an isomorphism. (Note that both  $\bigoplus_{\substack{\eta=(n_1, \dots, n_r) \\ n_1 + \dots + n_r = n}} \mathcal{S}_{n_1,1}^\eta(\Lambda_{n_1,1}(m_1)) \otimes \dots \otimes \mathcal{S}_{n_r,1}^\eta(\Lambda_{n_r,1}(m_r))$  and  $U_q(\mathfrak{g})$  are semi-simple.) (ii) follows from (i) by restricting  $\Phi_g$  to  ${}_{\mathcal{A}}U_q(\mathfrak{g})$ .  $\square$



**2.3.** For an  $\mathcal{S}_{n,r}$ -module  $M$ , we regard  $M$  as a  $U_q(\mathfrak{g})$ -module through the homomorphism  $\Phi_{\mathfrak{g}}$ . Then, by Lemma 2.2 (or by investigating weights directly), we see that a simple  $U_q(\mathfrak{g})$ -module appearing in  $M$  as a composition factor is the form  $W(\lambda^{(1)}) \boxtimes \cdots \boxtimes W(\lambda^{(r)})$  for some  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ , where  $W(\lambda^{(k)})$  is the highest weight  $U_q(\mathfrak{gl}_{m_k})$ -module of highest weight  $\lambda^{(k)}$ . Hence, the Weyl module  $W(\lambda)$  of  $\mathcal{S}_{n,r}$  decomposes as follows:

$$(2.3.1) \quad W(\lambda) \cong \bigoplus_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} \left( W(\mu^{(1)}) \boxtimes \cdots \boxtimes W(\mu^{(r)}) \right)^{\oplus \beta_{\lambda\mu}} \quad \text{as } U_q(\mathfrak{g})\text{-modules.}$$

**2.4.** In order to compute the multiplicity  $\beta_{\lambda\mu}$  in (2.3.1), we will describe the  $U_q(\mathfrak{g})$ -crystal structure on  $W(\lambda)$ . For such a purpose, we prepare some notation of combinatorics.

For  $\mu \in \Lambda_{n,r}(\mathbf{m})$ , the diagram  $[\mu]$  of  $\mu$  is the set  $\{(i, j, k) \in \mathbb{Z}^3 \mid 1 \leq i \leq m_k, 1 \leq j \leq \mu_i^{(k)}, 1 \leq k \leq r\}$ . For  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$  and  $\mu \in \Lambda_{n,r}(\mathbf{m})$ , a tableau of shape  $\lambda$  with weight  $\mu$  is a map  $T : [\lambda] \rightarrow \{(a, c) \in \mathbb{Z} \times \mathbb{Z} \mid a \geq 1, 1 \leq c \leq r\}$  such that  $\mu_i^{(k)} = \#\{x \in [\lambda] \mid T(x) = (i, k)\}$ . We define the order on  $\mathbb{Z} \times \mathbb{Z}$  by  $(a, c) \geq (a', c')$  if either  $c > c'$ , or  $c = c'$  and  $a \geq a'$ . For a tableau  $T$  of shape  $\lambda$  with weight  $\mu$ , we say that  $T$  is semi-standard if  $T$  satisfies the following conditions:

- (i) If  $T((i, j, k)) = (a, c)$ , then  $k \leq c$ ,
- (ii)  $T((i, j, k)) \leq T((i, j+1, k))$  if  $(i, j+1, k) \in [\lambda]$ ,
- (iii)  $T((i, j, k)) < T((i+1, j, k))$  if  $(i+1, j, k) \in [\lambda]$ .

For  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ ,  $\mu \in \Lambda_{n,r}(\mathbf{m})$ , we denote by  $\mathcal{T}_0(\lambda, \mu)$  the set of semi-standard tableaux of shape  $\lambda$  with weight  $\mu$ . Put  $\mathcal{T}_0(\lambda) = \bigcup_{\mu \in \Lambda_{n,r}(\mathbf{m})} \mathcal{T}_0(\lambda, \mu)$ . We identify a semi-standard tableau with a Young tableau as the following example.

For  $\lambda = ((3, 2), (3, 1), (1, 1))$ ,  $\mu = ((2, 1), (2, 2), (3, 1))$

$$T = \left( \begin{array}{|c|c|c|} \hline (1, 1) & (1, 1) & (1, 2) \\ \hline (2, 1) & (1, 3) & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline (1, 2) & (2, 2) & (1, 3) \\ \hline (2, 2) & & \\ \hline \end{array}, \begin{array}{|c|} \hline (1, 3) \\ \hline (2, 3) \\ \hline \end{array} \right) \in \mathcal{T}_0(\lambda, \mu),$$

where  $T((1, 1, 1)) = (1, 1)$ ,  $T((1, 2, 1)) = (1, 1)$ ,  $\dots$ ,  $T((2, 1, 3)) = (2, 3)$ .

By [DJM], it is known that there exists a bijection between  $\mathcal{T}_0(\lambda, \mu)$  and a basis of  $W(\lambda)_{\mu}$ . Hence, we will describe a  $U_q(\mathfrak{g})$ -crystal structure on  $\mathcal{T}_0(\lambda)$  which is isomorphic to the  $U_q(\mathfrak{g})$ -crystal on  $W(\lambda)$ .

**2.5.** By (2.3.1), for  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ ,  $\mu \in \Lambda_{n,r}(\mathbf{m})$ , we have

$$(2.5.1) \quad \begin{aligned} \#\mathcal{T}_0(\lambda, \mu) &= \dim W(\lambda)_{\mu} \\ &= \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\nu} \cdot \dim \left( W(\nu^{(1)}) \boxtimes \cdots \boxtimes W(\nu^{(r)}) \right)_{\mu} \\ &= \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r \dim W(\nu^{(k)})_{\mu^{(k)}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r \# \mathcal{T}_0(\nu^{(k)}, \mu^{(k)}) \\
&= \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r K_{\nu^{(k)}\mu^{(k)}},
\end{aligned}$$

where  $K_{\nu^{(k)}\mu^{(k)}}$  is the Kostka number. We have the following properties of  $\beta_{\lambda\mu}$ .

**Lemma 2.6.**

- (i) For  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ ,  $\beta_{\lambda\lambda} = 1$ .
- (ii) For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , if  $\beta_{\lambda\mu} \neq 0$ , we have  $\lambda \geq \mu$ .
- (iii) For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , if  $\lambda \neq \mu$  and  $\zeta(\lambda) = \zeta(\mu)$ , we have  $\beta_{\lambda\mu} = 0$ .
- (iv) For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta(\lambda) \neq \zeta(\mu)$ , if  $\mathcal{T}_0(\lambda, \nu) = \emptyset$  for any  $\nu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta(\nu) = \zeta(\mu)$  and  $\nu > \mu$ , then we have  $\beta_{\lambda\mu} = \# \mathcal{T}_0(\lambda, \mu)$ .

*Proof.* (i) From the definition of  $W(\lambda)$ , we have  $W(\lambda) = \mathcal{S}_{n,r}^- \cdot v_\lambda$ , where we denote  $1 \otimes v_\lambda \in \mathcal{S}_{n,r} \otimes_{\mathcal{S}_{n,r}^{\geq 0}} \theta_\lambda$  by  $v_\lambda$  simply. Thus, we have that  $W(\lambda)_\lambda = \mathcal{K}v_\lambda$ , and that  $v_\lambda$  is a highest weight vector of highest weight  $\lambda$  in  $U_q(\mathfrak{g})$ -module  $W(\lambda)$ . This implies that  $\beta_{\lambda\lambda} = 1$ .

(ii)  $\beta_{\lambda\mu} \neq 0 \Rightarrow W(\lambda)_\mu \neq 0 \Rightarrow \lambda \geq \mu$ .

(iii) Assume that  $\lambda \neq \mu$  and  $\zeta(\lambda) = \zeta(\mu)$ . By (2.5.1), we have

$$\begin{aligned}
(2.6.1) \quad \# \mathcal{T}_0(\lambda, \mu) &= \beta_{\lambda\lambda} \prod_{k=1}^r \# \mathcal{T}_0(\lambda^{(k)}, \mu^{(k)}) + \beta_{\lambda\mu} \prod_{k=1}^r \# \mathcal{T}_0(\mu^{(k)}, \mu^{(k)}) \\
&\quad + \sum_{\substack{\nu \in \Lambda_{n,r}^+(\mathbf{m}) \\ \nu \neq \lambda, \mu}} \beta_{\lambda\nu} \prod_{k=1}^r \# \mathcal{T}_0(\nu^{(k)}, \mu^{(k)}).
\end{aligned}$$

This implies that  $\beta_{\lambda\mu} = 0$  since  $\# \mathcal{T}_0(\mu^{(k)}, \mu^{(k)}) = 1$ , and  $\# \mathcal{T}_0(\lambda, \mu) = \prod_{k=1}^r \# \mathcal{T}_0(\lambda^{(k)}, \mu^{(k)})$  if  $\zeta(\lambda) = \zeta(\mu)$ .

(iv) Note that  $\prod_{k=1}^r \# \mathcal{T}_0(\nu^{(k)}, \mu^{(k)}) = 0$  if  $\zeta(\nu) \neq \zeta(\mu)$  or  $\nu \not\geq \mu$ , and that  $\prod_{k=1}^r \# \mathcal{T}_0(\nu^{(k)}, \mu^{(k)}) = \mathcal{T}_0(\nu, \mu)$  if  $\zeta(\nu) = \zeta(\mu)$ . Then (2.5.1) combining with the assumption of (iv) implies  $\# \mathcal{T}_0(\lambda, \mu) = \beta_{\lambda\mu} \# \mathcal{T}_0(\mu, \mu) = \beta_{\lambda\mu}$  since  $\beta_{\lambda\nu} = 0$  if  $\mathcal{T}_0(\lambda, \nu) = \emptyset$ .  $\square$

**2.7.** For  $\lambda \in \Lambda_{n,r}(\mathbf{m})$ , we define the total order “ $\succ$ ” on the diagram  $[\lambda]$  by  $(i, j, k) \succ (i', j', k')$  if  $k > k'$ ,  $k = k'$  and  $j > j'$  or if  $k = k'$ ,  $j = j'$  and  $i < i'$ . For an example, we have

$$(5, 4, 2) \succ (2, 3, 2) \succ (5, 3, 2) \succ (6, 4, 1).$$

**2.8.** We define the equivalence relation “ $\sim$ ” on  $\mathcal{T}_0(\lambda)$  by  $T \sim T'$  if  $\{x \in [\lambda] \mid T(x) = (i, k) \text{ for some } i = 1, \dots, m_k\} = \{y \in [\lambda] \mid T'(y) = (j, k) \text{ for some } j = 1, \dots, m_k\}$  for any  $k = 1, \dots, r$ . By the definition, for  $T \in \mathcal{T}_0(\lambda, \mu)$  and  $T' \in \mathcal{T}_0(\lambda, \nu)$ , we have

$$(2.8.1) \quad \zeta(\mu) = \zeta(\nu) \text{ if } T \sim T'.$$

**Example 2.9.** Put

$$T_1 = \left( \begin{array}{|c|c|} \hline (1,1) & (1,1) \\ \hline (1,2) & (2,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (1,2) & (2,2) \\ \hline (3,2) & \\ \hline \end{array} \right), \quad T_2 = \left( \begin{array}{|c|c|} \hline (1,1) & (2,1) \\ \hline (1,2) & (3,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (2,2) & (2,2) \\ \hline (4,2) & \\ \hline \end{array} \right),$$

$$T_3 = \left( \begin{array}{|c|c|} \hline (1,1) & (1,2) \\ \hline (2,1) & (3,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (2,2) & (2,2) \\ \hline (4,2) & \\ \hline \end{array} \right), \quad T_4 = \left( \begin{array}{|c|c|} \hline (1,1) & (2,2) \\ \hline (3,1) & (3,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (1,2) & (1,2) \\ \hline (2,2) & \\ \hline \end{array} \right).$$

Then, we have  $T_1 \sim T_2$ ,  $T_2 \not\sim T_3$  and  $T_3 \sim T_4$ .

**2.10.** Let  $V_{m_k}$  be the vector representation of  $U_q(\mathfrak{gl}_{m_k})$  with a natural basis  $\{v_1, v_2, \dots, v_{m_k}\}$ . Let  $\mathcal{A}_0$  be the localization of  $\mathbb{Q}(Q_1, \dots, Q_r)[q]$  at  $q = 0$ . Put  $\mathcal{L}_{m_k} = \bigoplus_{j=1}^{m_k} \mathcal{A}_0 \cdot v_j$ ,  $[j] = v_j + q\mathcal{L}_{m_k} \in \mathcal{L}_{m_k}/q\mathcal{L}_{m_k}$  and  $\mathcal{B}_{m_k} = \{[j] \mid 1 \leq j \leq m_k\}$ . Then  $(\mathcal{L}_{m_k}, \mathcal{B}_{m_k})$  gives the crystal basis of  $V_{m_k}$ . We denote by  $\mathcal{B}_{m_1}^{\otimes n_1} \boxtimes \dots \boxtimes \mathcal{B}_{m_r}^{\otimes n_r}$  the  $U_q(\mathfrak{g})$ -crystal corresponding to  $U_q(\mathfrak{g})$ -module  $V_{m_1}^{\otimes n_1} \boxtimes \dots \boxtimes V_{m_r}^{\otimes n_r}$ .

Let  $\mathcal{T}_0(\lambda) = \bigcup_t \mathcal{T}_0(\lambda)[t]$  be the decomposition to equivalence classes with respect to the equivalence relation “ $\sim$ ”. For an each equivalence class  $\mathcal{T}_0(\lambda)[t]$ , put  $(n_1, \dots, n_r) = \zeta(\mu)$  for some  $\mu$  such that  $\mathcal{T}_0(\lambda, \mu) \cap \mathcal{T}_0(\lambda)[t] \neq \emptyset$  (note (2.8.1)), and we define the map  $\Psi_t^\lambda : \mathcal{T}_0(\lambda)[t] \rightarrow \mathcal{B}_{m_1}^{\otimes n_1} \boxtimes \dots \boxtimes \mathcal{B}_{m_r}^{\otimes n_r}$  as

$$\Psi_t^\lambda(T) = \left( [i_1^{(1)}] \otimes \dots \otimes [i_{n_1}^{(1)}] \right) \boxtimes \dots \boxtimes \left( [i_1^{(r)}] \otimes \dots \otimes [i_{n_r}^{(r)}] \right)$$

satisfying the following three conditions:

- (i)  $\{x \in [\lambda] \mid T(x) = (i, k) \text{ for some } i = 1, \dots, m_k\} = \{x_1^{(k)}, x_2^{(k)}, \dots, x_{n_k}^{(k)}\}$  for  $k = 1, \dots, r$ .
- (ii)  $x_1^{(k)} \succ x_2^{(k)} \succ \dots \succ x_{n_k}^{(k)}$  for  $k = 1, \dots, r$ .
- (iii)  $T(x_j^{(k)}) = (i_j^{(k)}, k)$  for  $1 \leq j \leq n_k$ ,  $1 \leq k \leq r$ .

Namely,  $\left( [i_1^{(k)}] \otimes \dots \otimes [i_{n_k}^{(k)}] \right)$  in  $\Psi_t^\lambda(T)$  is obtained by reading the first coordinate of  $T(x)$  for  $x \in [\lambda]$  such that  $T(x) = (i, k)$  for some  $i = 1, \dots, m_k$  in the order “ $\succ$ ” on  $[\lambda]$ .

**Example 2.11.** For

$$T = \left( \begin{array}{|c|c|c|} \hline (1,1) & (1,1) & (1,2) \\ \hline (2,1) & (1,3) & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline (1,2) & (2,2) & (1,3) \\ \hline (2,2) & & \\ \hline \end{array}, \begin{array}{|c|} \hline (1,3) \\ \hline (2,3) \\ \hline \end{array} \right) \in \mathcal{T}_0(\lambda)[t],$$

we have  $\Psi_t^\lambda(T) = ([1] \otimes [1] \otimes [2]) \boxtimes ([2] \otimes [1] \otimes [2] \otimes [1]) \boxtimes ([1] \otimes [2] \otimes [1] \otimes [1])$ .

**Remark 2.12.** In the case where  $r = 1$ ,  $\mathcal{T}_0(\lambda)$  has only one equivalence class (itself) with respect to “ $\sim$ ”, and  $\Psi^\lambda$  coincides with the Far-Eastern reading given in [KN, §3] (see also [HK, Ch. 7]).

**2.13.** Let  $\tilde{e}_{(i,k)}, \tilde{f}_{(i,k)}$  ( $(i,k) \in I'_g(\mathbf{m})$ ) be the Kashiwara operators on  $U_q(\mathfrak{g})$ -crystal  $\mathcal{B}_{m_1}^{\otimes n_1} \boxtimes \dots \boxtimes \mathcal{B}_{m_r}^{\otimes n_r}$ . Then we have the following proposition.

**Proposition 2.14.** *For an each equivalence class  $\mathcal{T}_0(\lambda)[t]$  of  $\mathcal{T}_0(\lambda)$ , we have the followings.*

- (i) The map  $\Psi_t^\lambda : \mathcal{T}_0(\lambda)[t] \rightarrow \mathcal{B}_{m_k}^{\otimes n_1} \boxtimes \cdots \boxtimes \mathcal{B}_{m_k}^{\otimes n_r}$  is injective.
- (ii)  $\Psi_t^\lambda(\mathcal{T}_0(\lambda)[t]) \cup \{0\}$  is stable under the Kashiwara operators  $\tilde{e}_{(i,k)}, \tilde{f}_{(i,k)}$  ( $(i,k) \in \Gamma'_g(\mathbf{m})$ ).

*Proof.* (i) is clear from the definitions. (ii) is proven in a similar way as in the case of type  $A$  ( $r = 1$ ) (see [KN] or [HK, Theorem 7.3.6]).  $\square$

**2.15.** By Proposition 2.14, we define the  $U_q(\mathfrak{g})$ -crystal structure on  $\mathcal{T}_0(\lambda)[t]$  through  $\Psi_t^\lambda$ , and also define the  $U_q(\mathfrak{g})$ -crystal structure on  $\mathcal{T}_0(\lambda)$ . Note that the  $U_q(\mathfrak{g})$ -crystal graphs of  $\mathcal{T}_0(\lambda)[t]$  and of  $\mathcal{T}_0(\lambda)[t']$  are disconnected in the  $U_q(\mathfrak{g})$ -crystal graph of  $\mathcal{T}_0(\lambda)$  if  $\mathcal{T}_0(\lambda)[t]$  is a different equivalence class from  $\mathcal{T}_0(\lambda)[t']$ . For  $T \in \mathcal{T}_0(\lambda)$ , we say that  $T$  is  $U_q(\mathfrak{g})$ -singular if  $\tilde{e}_{(i,k)} \cdot T = 0$  for any  $(i,k) \in \Gamma'_g(\mathbf{m})$ .

**Remark 2.16.** We should define the map  $\Psi_t^\lambda$  for each equivalence class  $\mathcal{T}_0(\lambda)[t]$  of  $\mathcal{T}_0(\lambda)$  since it may happen that  $\Psi_t^\lambda(T) = \Psi_{t'}^\lambda(T')$  for different equivalence classes  $\mathcal{T}_0(\lambda)[t]$  and  $\mathcal{T}_0(\lambda)[t']$ . For an example, put

$$T = \left( \begin{array}{|c|c|c|} \hline (1,1) & (1,1) & (1,2) \\ \hline (2,1) & (1,3) & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline (1,2) & (2,2) & (1,3) \\ \hline (2,2) & & \\ \hline \end{array}, \begin{array}{|c|} \hline (1,3) \\ \hline (2,3) \\ \hline \end{array} \right) \in \mathcal{T}_0(\lambda)[t],$$

$$T' = \left( \begin{array}{|c|c|c|} \hline (1,1) & (1,1) & (1,3) \\ \hline (2,1) & (1,2) & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline (1,2) & (2,2) & (1,3) \\ \hline (2,2) & & \\ \hline \end{array}, \begin{array}{|c|} \hline (1,3) \\ \hline (2,3) \\ \hline \end{array} \right) \in \mathcal{T}_0(\lambda)[t'].$$

Then we have

$$\Psi_t^\lambda(T) = \Psi_{t'}^\lambda(T') = (\boxed{1} \otimes \boxed{1} \otimes \boxed{2}) \boxtimes (\boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1}) \boxtimes (\boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1}).$$

Now, we have the following theorem.

**Theorem 2.17.**

- (i) For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , we have

$$\beta_{\lambda\mu} = \sharp\{T \in \mathcal{T}_0(\lambda, \mu) \mid T : U_q(\mathfrak{g})\text{-singular}\}.$$

- (ii)  $U_q(\mathfrak{g})$ -crystal structure on  $\mathcal{T}_0(\lambda)$  is isomorphic to the  $U_q(\mathfrak{g})$ -crystal basis of  $W(\lambda)$  as crystals.

*Proof.* We prove (i) by an induction for dominance order “ $\geq$ ” on  $\Lambda_{n,r}^+(\mathbf{m})$ . First, we assume that  $\mathcal{T}_0(\lambda, \mu) \neq \emptyset$  and  $\mathcal{T}_0(\lambda, \nu) = \emptyset$  for any  $\nu$  such that  $\zeta(\nu) = \zeta(\mu)$  and  $\nu > \mu$ . For  $(i,k) \in \Gamma'_g(\mathbf{m})$ , one see easily that  $\tilde{e}_{(i,k)} \cdot T \in \mathcal{T}_0(\lambda, \mu + \alpha_{(i,k)})$ ,  $\zeta(\mu + \alpha_{(i,k)}) = \zeta(\mu)$  and  $\mu + \alpha_{(i,k)} > \mu$ . Then, for any  $T \in \mathcal{T}_0(\lambda, \mu)$  and any  $(i,k) \in \Gamma'_g(\mathbf{m})$ , we have  $\tilde{e}_{(i,k)} \cdot T = 0$  by the assumption. Thus, we have  $\sharp\mathcal{T}_0(\lambda, \mu) = \sharp\{T \in \mathcal{T}_0(\lambda, \mu) \mid T : U_q(\mathfrak{g})\text{-singular}\}$ . Combining with Lemma 2.6 (iv), we have  $\beta_{\lambda\mu} = \sharp\{T \in \mathcal{T}_0(\lambda, \mu) \mid T : U_q(\mathfrak{g})\text{-singular}\}$ .

Next, as the assumption of the induction, we assume the claim of (i) for  $\nu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta(\nu) = \zeta(\mu)$  and  $\nu > \mu$ . It is clear that

$$\dim W(\lambda)_\mu = \sum_{\substack{\nu \in \Lambda_{n,r}^+(\mathbf{m}) \\ \zeta(\nu) = \zeta(\mu), \nu \geq \mu}} \beta_{\lambda\nu} \cdot \dim (W(\nu^{(1)}) \boxtimes \cdots \boxtimes W(\nu^{(r)}))_\mu.$$

Thus, we have

$$(2.17.1) \quad \beta_{\lambda\mu} = \dim W(\lambda)_\mu - \sum_{\substack{\nu \in \Lambda_{n,r}^+(\mathbf{m}) \\ \zeta(\nu) = \zeta(\mu), \nu > \mu}} \beta_{\lambda\nu} \cdot \dim (W(\nu^{(1)}) \boxtimes \cdots \boxtimes W(\nu^{(r)}))_\mu.$$

If  $T \in \mathcal{T}_0(\lambda, \mu)$  is not  $U_q(\mathfrak{g})$ -singular, there exists a sequence  $(i_1, k_1), \dots, (i_l, k_l) \in \Gamma'_\mathfrak{g}(\mathbf{m})$  such that  $\tilde{e}_{(i_1, k_1)} \cdots \tilde{e}_{(i_l, k_l)} \cdot T \in \mathcal{T}_0(\lambda, \mu + \alpha_{(i_1, k_1)} + \cdots + \alpha_{(i_l, k_l)})$  is  $U_q(\mathfrak{g})$ -singular. Thus, by the assumption of the induction, we have

$$(2.17.2) \quad \begin{aligned} & \sum_{\substack{\nu \in \Lambda_{n,r}^+(\mathbf{m}) \\ \zeta(\nu) = \zeta(\mu), \nu > \mu}} \beta_{\lambda\nu} \cdot \dim (W(\nu^{(1)}) \boxtimes \cdots \boxtimes W(\nu^{(r)}))_\mu \\ &= \sharp \{ T \in \mathcal{T}_0(\lambda, \mu) \mid \tilde{e}_{(i,k)} \cdot T \neq 0 \text{ for some } (i, k) \in \Gamma'_\mathfrak{g} \}. \end{aligned}$$

Since  $\dim W(\lambda)_\mu = \mathcal{T}_0(\lambda, \mu)$ , (2.17.1) and (2.17.2) imply that  $\beta_{\lambda\mu} = \sharp \{ T \in \mathcal{T}_0(\lambda, \mu) \mid T : U_q(\mathfrak{g})\text{-singular} \}$ .

(ii) follows from (i) and the definition of  $\Psi_t^\lambda$ .  $\square$

### § 3. SOME PROPERTIES OF THE NUMBER $\beta_{\lambda\mu}$

In this section, we collect some properties of the number  $\beta_{\lambda\mu}$ . For some extreme partitions, we have the following lemma.

**Lemma 3.1.**

- (i) If  $\lambda = ((n), \emptyset, \dots, \emptyset)$ ,  

$$\beta_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu = ((n_1), (n_2), \dots, (n_r)) \text{ for some } (n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}^r. \\ 0 & \text{otherwise} \end{cases}$$
- (ii) If  $\lambda = ((1^n), \emptyset, \dots, \emptyset)$ ,  

$$\beta_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu = ((1^{n_1}), (1^{n_2}), \dots, (1^{n_r})) \text{ for some } (n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}^r. \\ 0 & \text{otherwise} \end{cases}$$
- (iii) If  $\mu = (\emptyset, \dots, \emptyset, (n))$ ,  

$$\beta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = ((n_1), (n_2), \dots, (n_r)) \text{ for some } (n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}^r. \\ 0 & \text{otherwise} \end{cases}$$
- (iv) If  $\mu = (\emptyset, \dots, \emptyset, (1^n))$ ,  

$$\beta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = ((1^{n_1}), (1^{n_2}), \dots, (1^{n_r})) \text{ for some } (n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}^r. \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* One can easily check them by using Lemma 3.3 and Theorem 2.17.  $\square$

**3.2.** For  $r$ -partitions  $\lambda$  and  $\mu$ , we denote by  $\lambda \supset \mu$  if  $[\lambda] \supset [\mu]$ . For  $r$ -partitions  $\lambda$  and  $\mu$  such that  $\lambda \supset \mu$ , we define the skew Young diagram by  $\lambda/\mu = [\lambda] \setminus [\mu]$ . One can naturally identify  $\lambda/\mu$  with  $(\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(r)}/\mu^{(r)})$ , where  $\lambda^{(k)}/\mu^{(k)}$  ( $1 \leq k \leq r$ ) is the usual skew Young diagram for  $\lambda^{(k)} \supset \mu^{(k)}$ . For a skew Young diagram  $\lambda/\mu$ , we define a semi-standard tableau of shape  $\lambda/\mu$  in a similar manner as in the case where the shape is an  $r$ -partition. We denote by  $\mathcal{T}_0(\lambda/\mu, \nu)$  the set of semi-standard tableaux of shape  $\lambda/\mu$  with weight  $\nu$ . Put  $\mathcal{T}_0(\lambda/\mu) = \bigcup_{\nu \in \Lambda_{n',r}(\mathbf{m})} \mathcal{T}_0(\lambda/\mu, \nu)$ , where  $n' = |\lambda/\mu|$ . Then, we can describe the  $U_q(\mathfrak{g})$ -crystal structure on  $\mathcal{T}_0(\lambda/\mu)$  in a similar way as in the paragraphs 2.7 - 2.15. Namely, we define the equivalence relation “ $\sim$ ” on  $\mathcal{T}_0(\lambda/\mu)$  in a similar way as in 2.8, and define the map  $\Psi_t^{\lambda/\mu} : \mathcal{T}_0(\lambda/\mu)[t] \rightarrow \mathcal{B}_{m_1}^{\otimes n_1} \boxtimes \dots \boxtimes \mathcal{B}_{m_k}^{\otimes n_k}$  for an each equivalence class  $\mathcal{T}_0(\lambda/\mu)[t]$  of  $\mathcal{T}_0(\lambda/\mu)$  as in 2.10. Then we can show that  $\Psi_t^{\lambda/\mu}$  is injective, and that  $\Psi_t^{\lambda/\mu}(\mathcal{T}_0(\lambda/\mu)[t]) \cup \{0\}$  is stable under the Kashiwara operators  $\tilde{e}_{(i,k)}, \tilde{f}_{(i,k)}$  for  $(i,k) \in I'_g(\mathbf{m})$  (cf. Proposition 2.14). Put

$$\mathcal{T}_{\text{sing}}(\lambda/\mu, \nu) = \{T \in \mathcal{T}_0(\lambda/\mu, \nu) \mid T : U_q(\mathfrak{g})\text{-singular}\}.$$

From the tensor product rule for  $U_q(\mathfrak{g})$ -crystals, we have the following criterion on whether  $T \in \mathcal{T}_0(\lambda/\mu)$  is  $U_q(\mathfrak{g})$ -singular or not (note that  $\mathcal{T}_0(\lambda/\mu) = \mathcal{T}_0(\lambda)$  if  $\mu = \emptyset$ ).

**Lemma 3.3.** *For  $T \in \mathcal{T}_0(\lambda/\mu)[t]$ , let  $\Psi_t^{\lambda/\mu}(T) = (\boxed{i_1^{(1)}} \otimes \dots \otimes \boxed{i_{n_1}^{(1)}}) \boxtimes \dots \boxtimes (\boxed{i_1^{(r)}} \otimes \dots \otimes \boxed{i_{n_r}^{(r)}})$ . Then,  $T$  is  $U_q(\mathfrak{g})$ -singular if and only if the weight of  $(\boxed{i_1^{(k)}} \otimes \dots \otimes \boxed{i_j^{(k)}}) \in \mathcal{B}_{m_k}^{\otimes j}$  is a partition (i.e. dominant integral weight of  $\mathfrak{gl}_{m_k}$ ) for any  $1 \leq j \leq n_k$  and any  $1 \leq k \leq r$ .*

*Proof.* It is clear that, for  $T \in \mathcal{T}_0(\lambda/\mu)[t]$  such that  $\Psi_t^{\lambda/\mu}(T) = (\boxed{i_1^{(1)}} \otimes \dots \otimes \boxed{i_{n_1}^{(1)}}) \boxtimes \dots \boxtimes (\boxed{i_1^{(r)}} \otimes \dots \otimes \boxed{i_{n_r}^{(r)}})$ ,  $T$  is  $U_q(\mathfrak{g})$ -singular if and only if  $(\boxed{i_1^{(k)}} \otimes \dots \otimes \boxed{i_{n_k}^{(k)}}) \in \mathcal{B}_{m_k}^{\otimes n_k}$  is  $U_q(\mathfrak{gl}_{m_k})$ -singular for any  $k = 1, \dots, r$ . Hence, the lemma follows from [N, Lemma 6.1.1] (see also [HK, Corollary 4.4.4]).  $\square$

**3.4.** Fix  $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}_{\geq 0}^g$  such that  $\sum_{k=1}^g r_k = r$ . For  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \Lambda_{n,r}^+(\mathbf{m})$ , put  $\lambda^{[k]\mathbf{p}} = (\lambda^{(p_k+1)}, \dots, \lambda^{(p_k+r_k)})$ , where  $p_k = \sum_{j=1}^{k-1} r_j$  with  $p_1 = 0$ . We define the map  $\zeta^{\mathbf{p}} : \Lambda_{n,r}^+(\mathbf{m}) \rightarrow \mathbb{Z}_{\geq 0}^g$  by  $\zeta^{\mathbf{p}}(\lambda) = (|\lambda^{[1]\mathbf{p}}|, \dots, |\lambda^{[g]\mathbf{p}}|)$ . Then, we have the following lemma.

**Lemma 3.5.** *For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta^{\mathbf{p}}(\lambda) = \zeta^{\mathbf{p}}(\mu)$ , we have*

$$\beta_{\lambda\mu} = \prod_{k=1}^g \beta_{\lambda^{[k]\mathbf{p}}\mu^{[k]\mathbf{p}}}.$$

*Proof.* It is enough to show the case where  $\mathbf{p} = (r_1, r_2)$  since we can obtain the claim for general cases by the induction on  $g$ . If  $\zeta^{\mathbf{p}}(\lambda) = \zeta^{\mathbf{p}}(\mu)$  for  $\mathbf{p} = (r_1, r_2)$ , then we have the bijection

$$(3.5.1) \quad \mathcal{T}_0(\lambda, \mu) \rightarrow \mathcal{T}_0(\lambda^{[1]\mathbf{p}}, \mu^{[1]\mathbf{p}}) \times \mathcal{T}_0(\lambda^{[2]\mathbf{p}}, \mu^{[2]\mathbf{p}}) \text{ such that } T \mapsto (T^{[1]\mathbf{p}}, T^{[2]\mathbf{p}}),$$



where  $T^{[1]\mathbf{p}}((i, j, k)) = T((i, j, k))$  for  $(i, j, k) \in [\lambda^{[1]\mathbf{p}}]$ , and  $T^{[2]\mathbf{p}}((i, j, k)) = (a, c - r_1)$  if  $T((i, j, r_1 + k)) = (a, c)$  for  $(i, j, k) \in [\lambda^{[2]\mathbf{p}}]$ . In this case, by the definition of  $\Psi_t^\lambda$  and Lemma 3.3, it is clear that  $T \in \mathcal{T}_0(\lambda, \mu)$  is  $U_q(\mathfrak{g})$ -singular if and only if  $T^{[1]\mathbf{p}}$  (resp.  $T^{[2]\mathbf{p}}$ ) is  $U_q(\mathfrak{g}^{[1]})$ -singular (resp.  $U_q(\mathfrak{g}^{[2]})$ -singular), where  $\mathfrak{g}^{[1]} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_{r_1}}$  (resp.  $\mathfrak{g}^{[2]} = \mathfrak{gl}_{m_{r_1}+1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ ). Then, by Theorem 2.17 (i) together with (3.5.1), we have  $\beta_{\lambda\mu} = \beta_{\lambda^{[1]\mathbf{p}}\mu^{[1]\mathbf{p}}} \beta_{\lambda^{[2]\mathbf{p}}\mu^{[2]\mathbf{p}}}$ .  $\square$

**3.6.** For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , we define the following set of sequences of  $r$ -partitions:

$$\Theta(\lambda, \mu) := \left\{ \lambda = \lambda_{\langle r \rangle} \supset \lambda_{\langle r-1 \rangle} \supset \cdots \supset \lambda_{\langle 1 \rangle} \supset \lambda_{\langle 0 \rangle} = (\emptyset, \dots, \emptyset) \right. \\ \left. \mid (\lambda_{\langle k \rangle})^{(k+1)} = \emptyset, \quad |\lambda_{\langle k \rangle} / \lambda_{\langle k-1 \rangle}| = |\mu^{(k)}| \text{ for } k = 1, \dots, r \right\}.$$

It is clear that, for  $\lambda_{\langle r \rangle} \supset \cdots \supset \lambda_{\langle 0 \rangle} \in \Theta(\lambda, \mu)$ , we have that  $\lambda_{\langle k \rangle} = (\lambda_{\langle k \rangle}^{(1)}, \dots, \lambda_{\langle k \rangle}^{(k)}, \emptyset, \dots, \emptyset)$ , and that  $|\lambda_{\langle k \rangle}| = \sum_{j=1}^k |\mu^{(j)}|$ . Then, we can rewrite Theorem 2.17 (i) as the following corollary.

**Corollary 3.7.** *For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , we have*

$$(3.7.1) \quad \beta_{\lambda\mu} = \sum_{\lambda_{\langle r \rangle} \supset \cdots \supset \lambda_{\langle 0 \rangle} \in \Theta(\lambda, \mu)} \prod_{k=1}^r \# \mathcal{T}_{\text{sing}}(\lambda_{\langle k \rangle} / \lambda_{\langle k-1 \rangle}, (\emptyset, \dots, \emptyset, \mu^{(k)}, \emptyset, \dots, \emptyset)).$$

*In particular, if  $\lambda = (\emptyset, \dots, \emptyset, \lambda^{(t)}, \emptyset, \dots, \emptyset)$  for some  $t$ , then we have*

$$(3.7.2) \quad \beta_{\lambda\mu} = \sum_{\lambda_{\langle r \rangle} \supset \cdots \supset \lambda_{\langle 0 \rangle} \in \Theta(\lambda, \mu)} \prod_{k=1}^r \text{LR}_{\lambda_{\langle k-1 \rangle}^{(t)}, \mu^{(k)}}^{\lambda_{\langle k \rangle}^{(t)}},$$

where  $\text{LR}_{\lambda_{\langle k-1 \rangle}^{(t)}, \mu^{(k)}}^{\lambda_{\langle k \rangle}^{(t)}}$  is the Littlewood-Richardson coefficient for  $\lambda_{\langle k-1 \rangle}^{(t)}$ ,  $\mu^{(k)}$  and  $\lambda_{\langle k \rangle}^{(t)}$  with  $\text{LR}_{\emptyset, \emptyset}^{\emptyset} = 1$ .

*Proof.* Note that we can identify the set  $\Theta(\lambda, \mu)$  with the set of equivalence classes of  $\mathcal{T}_0(\lambda, \mu)$  with respect to the relation “ $\sim$ ” by corresponding  $\lambda_{\langle r \rangle} \supset \cdots \supset \lambda_{\langle 0 \rangle} \in \Theta(\lambda, \mu)$  to the equivalence class of  $\mathcal{T}_0(\lambda, \mu)$  containing  $T \in \mathcal{T}_0(\lambda, \mu)$  such that  $[\lambda_{\langle k \rangle}] = \{(i, j, l) \in [\lambda] \mid T((i, j, l)) = (a, c) \text{ for some } 1 \leq a \leq m_c, 1 \leq c \leq k\}$  for any  $k = 1, \dots, r$ . Then Lemma 3.3 and Theorem 2.17 (i) imply the equation (3.7.1).

Assume that  $\lambda = (\emptyset, \dots, \emptyset, \lambda^{(t)}, \emptyset, \dots, \emptyset)$  for some  $t$ . Then, for  $\lambda_{\langle r \rangle} \supset \cdots \supset \lambda_{\langle 0 \rangle} \in \Theta(\lambda, \mu)$ , we have

$$\begin{aligned} \# \mathcal{T}_{\text{sing}}(\lambda_{\langle k \rangle} / \lambda_{\langle k-1 \rangle}, (\emptyset, \dots, \emptyset, \mu^{(k)}, \emptyset, \dots, \emptyset)) &= \# \mathcal{T}_{\text{sing}}(\lambda_{\langle k \rangle}^{(t)} / \lambda_{\langle k-1 \rangle}^{(t)}, \mu^{(k)}) \\ &= \text{LR}_{\lambda_{\langle k-1 \rangle}^{(t)}, \mu^{(k)}}^{\lambda_{\langle k \rangle}^{(t)}}, \end{aligned}$$

where the last equation follows from the original Littlewood-Richardson rule ([Mac, Ch. I (9.2)]). (Note that, for partitions  $\lambda, \mu$  (not multi-partitions) such that  $\lambda \supset \mu$ , the  $U_q(\mathfrak{gl}_m)$ -crystal structure on  $\mathcal{T}_0(\lambda/\mu)$  does not depend on the choice of admissible reading (see [HK, Theorem 7.3.6]). Then a similar statement as in Lemma 3.3 for  $\mathcal{T}_0(\lambda/\mu)$  under the Middle-Eastern reading coincides with the Littlewood-Richardson rule.) Then (3.7.1) implies (3.7.2).  $\square$

**Remark 3.8.** In the case where  $r = 2$  and  $\lambda = (\lambda^{(1)}, \emptyset)$ , by (3.7.2), we have

$$\begin{aligned} \beta_{\lambda\mu} &= \sum_{\lambda_{(1)}^{(1)}} \text{LR}_{\lambda_{(1)}^{(1)}, \mu^{(1)}}^{\lambda_{(1)}^{(1)}} \text{LR}_{\emptyset, \mu^{(2)}}^{\lambda_{(1)}^{(1)}} \\ &= \text{LR}_{\mu^{(2)}, \mu^{(1)}}^{\lambda_{(1)}^{(1)}}, \end{aligned}$$

where the last equation follows from  $\text{LR}_{\emptyset, \mu^{(2)}}^{\lambda_{(1)}^{(1)}} = \delta_{\lambda_{(1)}^{(1)}, \mu^{(2)}}$ . Thus, the Littlewood-Richardson coefficient  $\text{LR}_{\mu, \nu}^{\lambda}$  for partitions  $\lambda, \mu, \nu$  is obtained as the number  $\beta_{(\lambda, \emptyset)(\mu, \nu)}$ . Moreover, thanks to Lemma 3.3 together with the reading  $\Psi_t^{\lambda/\mu}$ , we can regard (3.7.1) as a generalization of the Littlewood-Richardson rule. We also remark the following classical fact:

$$(3.8.1) \quad [\text{Res}_{\mathbf{GL}_m \times \mathbf{GL}_{n-m}}^{\mathbf{GL}_n} V_{\lambda} : V_{\mu} \boxtimes V_{\nu}]_{\mathbf{GL}_m \times \mathbf{GL}_n} = \text{LR}_{\mu, \nu}^{\lambda},$$

where  $\mathbf{GL}_n$  (resp.  $\mathbf{GL}_m, \mathbf{GL}_{n-m}$ ) is the general linear group of rank  $n$  (resp.  $m, n-m$ ), and  $V_{\lambda}$  (resp.  $V_{\mu}, V_{\nu}$ ) is the simple  $\mathbf{GL}_n$ -module (resp. simple  $\mathbf{GL}_m$ -module, simple  $\mathbf{GL}_{n-m}$ -module) corresponding to a partition  $\lambda$  (resp.  $\mu, \nu$ ). Comparing (2.3.1) with (3.8.1), we may regard the number  $\beta_{\lambda\mu}$  as a generalization of Littlewood-Richardson coefficients.

## § 4. CHARACTERS OF THE WEYL MODULES AND SYMMETRIC FUNCTIONS

**4.1.** For  $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{>0}^r$ , we denote by  $\Xi_{\mathbf{m}} = \bigotimes_{k=1}^r \mathbb{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}]^{\mathfrak{S}_{m_k}}$  the ring of symmetric polynomials (with respect to  $\mathfrak{S}_{m_1} \times \dots \times \mathfrak{S}_{m_r}$ ) with variables  $x_i^{(k)}$  ( $1 \leq i \leq m_k, 1 \leq k \leq r$ ). We denote by  $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_{m_k}^{(k)})$  the set of  $m_k$  independent variables for  $k = 1, \dots, r$ , and denote by  $\mathbf{x} = (x^{(1)}, \dots, x^{(r)})$  the whole variables. Let  $\Xi_{\mathbf{m}}^n$  be the subset of  $\Xi_{\mathbf{m}}$  which consists of homogeneous symmetric polynomials of degree  $n$ . We also consider the inverse limit  $\Xi^n = \varprojlim_{\mathbf{m}} \Xi_{\mathbf{m}}^n$  with respect to  $\mathbf{m}$ . Put  $\Xi = \bigoplus_{n \geq 0} \Xi^n$ . Then  $\Xi$  becomes the ring of symmetric functions  $\Xi = \bigotimes_{k=1}^r \mathbb{Z}[X^{(k)}]^{\mathfrak{S}(X^{(k)})}$ , where  $X^{(k)} = (X_1^{(k)}, X_2^{(k)}, \dots)$  is the set of (infinite) variables. We denote by  $\mathbf{X} = (X^{(1)}, \dots, X^{(r)})$  the whole variables of  $\Xi$ .

For  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \Lambda_{n, r}^+(\mathbf{m})$ , put  $S_{\lambda}(\mathbf{x}) = \prod_{k=1}^r S_{\lambda^{(k)}}(x^{(k)})$  (resp.  $S_{\lambda}(\mathbf{X}) = \prod_{k=1}^r S_{\lambda^{(k)}}(X^{(k)})$ ), where  $S_{\lambda^{(k)}}(x^{(k)})$  (resp.  $S_{\lambda^{(k)}}(X^{(k)})$ ) is the Schur polynomial (resp. Schur function) associated to  $\lambda^{(k)}$  ( $1 \leq k \leq r$ ) with variables  $x^{(k)}$  (resp.  $X^{(k)}$ ). Then

$\{S_\lambda(\mathbf{x}) \mid \lambda \in \Lambda_{n,r}^+(\mathbf{m})\}$  (resp.  $\{S_\lambda(\mathbf{X}) \mid \lambda \in \Lambda_{n,r}^+\}$ ) gives a  $\mathbb{Z}$ -basis of  $\Xi_{\mathbf{m}}^n$  (resp.  $\mathbb{Z}$ -basis of  $\Xi^n$ ).

**4.2.** For an  $\mathcal{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$ -module  $M$ , we define the character of  $M$  by

$$\text{ch } M = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \dim M_\mu \cdot x^\mu \in \mathbb{Z}[\mathbf{x}],$$

where  $x^\mu = \prod_{k=1}^r (x_1^{(k)})^{\mu_1^{(k)}} (x_2^{(k)})^{\mu_2^{(k)}} \cdots (x_{m_k}^{(k)})^{\mu_{m_k}^{(k)}}$ . Then the character of the Weyl module  $W(\lambda)$  for  $\mathcal{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$  has the following properties.

**Theorem 4.3.**

(i) For  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ , we have

$$\text{ch } W(\lambda) = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \left( \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r K_{\nu^{(k)}\mu^{(k)}} \right) \cdot x^\mu,$$

where  $K_{\nu^{(k)}\mu^{(k)}}$  is the Kostka number corresponding to partitions  $\nu^{(k)}$  and  $\mu^{(k)}$ .

(ii) Put  $\tilde{S}_\lambda(\mathbf{x}) = \text{ch } W(\lambda)$  for  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ . Then, we have

$$\tilde{S}_\lambda(\mathbf{x}) = \sum_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\mu} S_\mu(\mathbf{x}).$$

(iii)  $\{\tilde{S}_\lambda(\mathbf{x}) \mid \lambda \in \Lambda_{n,r}^+(\mathbf{m})\}$  gives a  $\mathbb{Z}$ -basis of  $\Xi_{\mathbf{m}}^n$ .

*Proof.* Note Remark 1.6, we may assume that  $m_k \geq n$  for any  $k = 1, \dots, r$  by restricting the weights if necessary for a general case.

Since there exists a bijection between a basis of  $W(\lambda)_\mu$  and  $\mathcal{T}_0(\lambda, \mu)$ , (i) follows from (2.5.1).

It is known that

$$(4.3.1) \quad S_\lambda(\mathbf{x}) = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \dim (W(\lambda^{(1)}) \boxtimes \cdots \boxtimes W(\lambda^{(r)}))_\mu \cdot x^\mu.$$

Note that the  $\mu$ -weight space of an  $\mathcal{S}_{n,r}$ -module coincides with the  $\mu$ -weight space as the  $U_q(\mathfrak{g})$ -module via the homomorphism  $\Phi_{\mathfrak{g}} : U_q(\mathfrak{g}) \rightarrow \mathcal{S}_{n,r}$ . Thus, the decomposition (2.3.1) together with (4.3.1) implies (ii).

(iii) follows from (ii) since the number  $\beta_{\lambda\mu}$  ( $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ ) has the uni-triangular property by Lemma 2.6.  $\square$

**4.4.** For  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ , let  $\tilde{S}_\lambda(\mathbf{X}) \in \Xi^n$  be the image of  $\tilde{S}_\lambda(\mathbf{x})$  in the inverse limit. We denote by  $\Lambda_{\geq 0, r}^+ = \bigcup_{n \geq 0} \Lambda_{n,r}^+$  the set of  $r$ -partitions. Then, Theorem 4.3 (iii) implies that  $\{\tilde{S}_\lambda(\mathbf{X}) \mid \lambda \in \Lambda_{\geq 0, r}^+\}$  gives a  $\mathbb{Z}$ -basis of  $\Xi$ . For a certain extreme  $r$ -partition  $\lambda$ ,  $\tilde{S}_\lambda(\mathbf{X})$  coincides with the Schur function as follows.

**Proposition 4.5.** For  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \Lambda_{n,r}^+$ , if  $\lambda^{(l)} = \emptyset$  unless  $l = t$  for some  $t$ , we have

$$\tilde{S}_\lambda(\mathbf{X}) = S_{\lambda^{(t)}}(X^{(t)} \cup X^{(t+1)} \cup \dots \cup X^{(r)}),$$

where  $S_{\lambda^{(t)}}(X^{(t)} \cup \dots \cup X^{(r)}) \in \mathbb{Z}[X^{(t)} \cup \dots \cup X^{(r)}]^{\mathfrak{S}(X^{(t)} \cup \dots \cup X^{(r)})}$  is the Schur function corresponding to the partition  $\lambda^{(t)}$ .

*Proof.* Assume that  $\lambda^{(l)} = \emptyset$  unless  $l = t$ , then we see that the variable  $X_i^{(l)}$  ( $i \geq 1$ ,  $1 \leq l \leq t-1$ ) does not appear in  $\tilde{S}_\lambda(\mathbf{X})$  since  $\lambda \geq \mu$  if  $\dim W(\lambda)_\mu \neq 0$ . Note that we can regard  $\mathbb{Z}[X^{(t)} \cup \dots \cup X^{(r)}]^{\mathfrak{S}(X^{(t)} \cup \dots \cup X^{(r)})}$  as a subring of  $\Xi = \bigotimes_{k=1}^r \mathbb{Z}[X^{(k)}]^{\mathfrak{S}(X^{(k)})}$  in the natural way. By Theorem 4.3 (ii) with (3.7.2), we have

$$\begin{aligned} \tilde{S}_\lambda(\mathbf{X}) &= \sum_{\mu \in \Lambda_{n,r}^+} \left( \sum_{\lambda_{(r)} \supset \dots \supset \lambda_{(0)} \in \Theta(\lambda, \mu)} \prod_{k=1}^r \text{LR}_{\lambda_{(k-1)}^{(t)}, \mu^{(k)}}^{\lambda_{(k)}^{(t)}} \right) S_\mu(\mathbf{X}) \\ &= \sum_{\mu \in \Lambda_{n,r}^+} \sum_{(*1)} \left( \prod_{k=t}^r \text{LR}_{\lambda_{(k-1)}^{(t)}, \mu^{(k)}}^{\lambda_{(k)}^{(t)}} S_{\mu^{(k)}}(X^{(k)}) \right) \\ &= \sum_{(*2)} \sum_{\mu \in \Lambda_{n,r}^+} \left( \prod_{k=t}^r \text{LR}_{\lambda_{(k-1)}^{(t)}, \mu^{(k)}}^{\lambda_{(k)}^{(t)}} S_{\mu^{(k)}}(X^{(k)}) \right) \\ &= \sum_{(*2)} \prod_{k=t}^r \left( \sum_{(*3)} \text{LR}_{\lambda_{(k-1)}^{(t)}, \mu^{(k)}}^{\lambda_{(k)}^{(t)}} S_{\mu^{(k)}}(X^{(k)}) \right) \\ &= \sum_{(*2)} \prod_{k=t}^r S_{\lambda_{(k)}^{(t)}/\lambda_{(k-1)}^{(t)}}(X^{(k)}) \quad (\text{because of [Mac, Ch. 1. (5.3)]}) \\ &= S_{\lambda^{(t)}}(X^{(t)} \cup X^{(t+1)} \cup \dots \cup X^{(r)}) \quad (\text{because of [Mac, Ch. 1. (5.11)]}), \end{aligned}$$

where the summations (\*1)-(\*3) run the following sets respectively:

$$\begin{aligned} (*1) : & \{ \lambda^{(t)} = \lambda_{(r)}^{(t)} \supset \dots \supset \lambda_{(t)}^{(t)} \supset \lambda_{(t-1)}^{(t)} = \emptyset \mid |\lambda_{(k)}^{(t)}/\lambda_{(k-1)}^{(t)}| = |\mu^{(k)}| \text{ for } k = t, \dots, r \}, \\ (*2) : & \{ \lambda^{(t)} = \lambda_{(r)}^{(t)} \supset \dots \supset \lambda_{(t)}^{(t)} \supset \lambda_{(t-1)}^{(t)} = \emptyset \}, \\ (*3) : & \{ \mu^{(k)} : \text{partition} \}. \end{aligned}$$

(In the above equations, note that  $\text{LR}_{\lambda_{(k-1)}^{(t)}, \mu^{(k)}}^{\lambda_{(k)}^{(t)}} = 0$  unless  $|\lambda_{(k)}^{(t)}| = |\lambda_{(k-1)}^{(t)}| + |\mu^{(k)}|$ .)  $\square$

**4.6.** Thanks to the above lemma, the symmetric function  $\tilde{S}_\lambda(\mathbf{X})$  seems a generalization of the Schur function.

For  $\lambda, \mu, \nu \in \Lambda_{\geq 0, r}^+$ , we define the integer  $c_{\lambda\mu}^\nu \in \mathbb{Z}$  by

$$\tilde{S}_\lambda(\mathbf{X})\tilde{S}_\mu(\mathbf{X}) = \sum_{\nu \in \Lambda_{\geq 0, r}^+} c_{\lambda\mu}^\nu \tilde{S}_\nu(\mathbf{X}).$$

Then we can compute the number  $c_{\lambda\mu}^\nu$  as follows.

**Proposition 4.7.** *For  $\lambda, \mu, \nu \in \Lambda_{\geq 0, r}^+$ , we have the following.*

- (i)  $c_{\lambda\mu}^\nu = 0$  unless  $|\nu| = |\lambda| + |\mu|$ .
- (ii) Put  $(\beta'_{\tau\nu})_{\tau, \nu \in \Lambda_{n, r}^+} = (\beta_{\tau\nu})_{\tau, \nu \in \Lambda_{n, r}^+}^{-1}$  ( $n = |\nu|$ ). Then we have

$$c_{\lambda\mu}^\nu = \sum_{\xi, \eta, \tau \in \Lambda_{\geq 0, r}} \beta_{\lambda\xi} \beta_{\mu\eta} \beta'_{\tau\nu} \prod_{k=1}^r \text{LR}_{\xi^{(k)}\eta^{(k)}}^{\tau^{(k)}}.$$

- (iii) If  $\zeta(\nu) = \zeta(\lambda + \mu)$ , we have

$$c_{\lambda\mu}^\nu = \prod_{k=1}^r \text{LR}_{\lambda^{(k)}\mu^{(k)}}^{\nu^{(k)}}.$$

- (iv) If  $\lambda^{(l)} = \emptyset$  and  $\mu^{(l)} = \emptyset$  unless  $l = t$  for some  $t$ , we have

$$c_{\lambda\mu}^\nu = \begin{cases} \text{LR}_{\lambda^{(t)}\mu^{(t)}}^{\nu^{(t)}} & \text{if } \nu^{(l)} = \emptyset \text{ unless } l = t, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* (i) is clear from the definitions. We prove (ii). By Theorem 4.3 (ii), we have

$$\begin{aligned} (4.7.1) \quad \tilde{S}_\lambda(\mathbf{X})\tilde{S}_\mu(\mathbf{X}) &= \left( \sum_{\xi} \beta_{\lambda\xi} S_\xi(\mathbf{X}) \right) \left( \sum_{\eta} \beta_{\mu\eta} S_\eta(\mathbf{X}) \right) \\ &= \sum_{\xi, \eta} \beta_{\lambda\xi} \beta_{\mu\eta} S_\xi(\mathbf{X}) S_\eta(\mathbf{X}) \\ &= \sum_{\xi, \eta} \beta_{\lambda\xi} \beta_{\mu\eta} \left( \sum_{\tau} \left( \prod_{k=1}^r \text{LR}_{\xi^{(k)}\eta^{(k)}}^{\tau^{(k)}} \right) S_\tau(\mathbf{X}) \right) \\ &= \sum_{\xi, \eta} \beta_{\lambda\xi} \beta_{\mu\eta} \left( \sum_{\tau} \left( \prod_{k=1}^r \text{LR}_{\xi^{(k)}\eta^{(k)}}^{\tau^{(k)}} \right) \left( \sum_{\nu} \beta'_{\tau\nu} \tilde{S}_\nu(\mathbf{X}) \right) \right) \\ &= \sum_{\nu} \left( \sum_{\xi, \eta, \tau} \beta_{\lambda\xi} \beta_{\mu\eta} \beta'_{\tau\nu} \prod_{k=1}^r \text{LR}_{\xi^{(k)}\eta^{(k)}}^{\tau^{(k)}} \right) \tilde{S}_\nu(\mathbf{X}). \end{aligned}$$

This implies (ii).

By Lemma 2.6 and the fact that  $\text{LR}_{\xi^{(k)}\eta^{(k)}}^{\nu^{(k)}} = 0$  unless  $|\nu^{(k)}| = |\xi^{(k)}| + |\eta^{(k)}|$ , the equations (4.7.1) imply that

$$\begin{aligned} & \tilde{S}_\lambda(\mathbf{X})\tilde{S}_\mu(\mathbf{X}) \\ &= \sum_{\substack{\nu \\ \zeta(\nu)=\zeta(\lambda+\mu)}} \left( \prod_{k=1}^r \text{LR}_{\lambda^{(k)}\mu^{(k)}}^{\nu^{(k)}} \right) S_\nu(\mathbf{X}) + \sum_{\substack{\nu \\ \zeta(\nu) < \zeta(\lambda+\mu)}} \left( \sum_{\xi, \eta} \beta_{\lambda\xi} \beta_{\mu\eta} \prod_{k=1}^r \text{LR}_{\xi^{(k)}\eta^{(k)}}^{\nu^{(k)}} \right) S_\nu(\mathbf{X}) \\ &= \sum_{\substack{\nu \\ \zeta(\nu)=\zeta(\lambda+\mu)}} \left( \prod_{k=1}^r \text{LR}_{\lambda^{(k)}\mu^{(k)}}^{\nu^{(k)}} \right) \tilde{S}_\nu(\mathbf{X}) + \sum_{\substack{\nu \\ \zeta(\nu) < \zeta(\lambda+\mu)}} a_{\lambda\mu}^\nu \tilde{S}_\nu(\mathbf{X}) \quad (a_{\lambda\mu}^\nu \in \mathbb{Z}). \end{aligned}$$

This implies (iii).

Finally, we prove (iv). By Proposition 4.5, we have

$$\begin{aligned} \tilde{S}_\lambda(\mathbf{X})\tilde{S}_\mu(\mathbf{X}) &= S_{\lambda^{(t)}}(X^{(t)} \cup \dots \cup X^{(r)}) S_{\mu^{(t)}}(X^{(t)} \cup \dots \cup X^{(r)}) \\ &= \sum_{\nu^{(t)}} \text{LR}_{\lambda^{(t)}\mu^{(t)}}^{\nu^{(t)}} S_{\nu^{(t)}}(X^{(t)} \cup \dots \cup X^{(r)}) \\ &= \sum_{\nu^{(t)}} \text{LR}_{\lambda^{(t)}\mu^{(t)}}^{\nu^{(t)}} \tilde{S}_{(\emptyset, \dots, \emptyset, \nu^{(t)}, \emptyset, \dots, \emptyset)}(\mathbf{X}). \end{aligned}$$

This implies (iv). □

**4.8.** We have some conjectures for the number  $c_{\lambda\mu}^\nu$  as follows.

**Conjecture 1:** For  $\lambda, \mu, \nu \in \Lambda_{\geq 0}^+$ , the number  $c_{\lambda\mu}^\nu$  is a non-negative integer.

More strongly, we conjecture the following.

**Conjecture 2:**  $c_{\lambda\mu}^\nu = \prod_{k=1}^r \text{LR}_{\lambda^{(k)}\mu^{(k)}}^{\nu^{(k)}}.$

Note that  $\text{LR}_{\lambda^{(k)}\mu^{(k)}}^{\nu^{(k)}} = 0$  if  $|\nu^{(k)}| \neq |\lambda^{(k)}| + |\mu^{(k)}|$ , then Conjecture 2 is equivalent to  $c_{\lambda\mu}^\nu = 0$  unless  $\zeta(\nu) = \zeta(\lambda + \mu)$  by Proposition 4.7 (iii).

We remark that Conjecture 2 is true for  $\lambda, \mu \in \Lambda_{\geq 0, r}^+$  such that  $\lambda^{(l)} = \emptyset$  and  $\mu^{(l)} = \emptyset$  unless  $l = t$  for some  $t$  by Proposition 4.7 (iv).

## § 5. DECOMPOSITION MATRICES OF CYCLOTOMIC $q$ -SCHUR ALGEBRAS

In this section, we consider the specialized cyclotomic  $q$ -Schur algebra  ${}_F\mathcal{S}_{n,r}$  over a field  $F$  with parameters  $q, Q_1, \dots, Q_r \in F$  such that  $q \neq 0$ . Hence, we omit the subscript  $F$  for the objects over  $F$ . We also denote by  $U_q(\mathfrak{g}) = F \otimes_{\mathcal{A}} {}_{\mathcal{A}}U_q(\mathfrak{g})$  simply. Through this section, we assume that  $m_k \geq n$  for any  $k = 1, \dots, r$ .

**5.1.** For  $\mathcal{S}_{n,r}$ -module  $M$ , we regard  $M$  as a  $U_q(\mathfrak{g})$ -module through the homomorphism  $\Phi_{\mathfrak{g}}$ . Then, by Lemma 2.2 (ii), we see that a simple  $U_q(\mathfrak{g})$ -module appearing in the composition series of  $M$  is the form  $L(\lambda^{(1)}) \boxtimes \dots \boxtimes L(\lambda^{(r)})$  ( $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ ), where  $L(\lambda^{(k)})$  is the simple  $U_q(\mathfrak{gl}_{m_k})$ -module with highest weight  $\lambda^{(k)}$ .



For a simple  $\mathcal{S}_{n,r}$ -module  $L(\lambda)$  ( $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ ), let

$$x_{\lambda\mu} = [L(\lambda) : L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})]_{U_q(\mathfrak{g})}$$

be the multiplicity of  $L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})$  ( $\mu \in \Lambda_{n,r}^+(\mathbf{m})$ ) in the composition series of  $L(\lambda)$  as  $U_q(\mathfrak{g})$ -modules through  $\Phi_{\mathfrak{g}}$ . Then we have the following lemma.

**Lemma 5.2.**

- (i) For  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ ,  $x_{\lambda\lambda} = 1$ .
- (ii) For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , if  $x_{\lambda\mu} \neq 0$ , we have  $\lambda \geq \mu$ .
- (iii) For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , if  $\lambda \neq \mu$  and  $\zeta(\lambda) = \zeta(\mu)$ , we have  $x_{\lambda\mu} = 0$ .

*Proof.* By the definition of Weyl modules (see 1.7), we have  $W(\lambda) = \mathcal{S}_{n,r}^- \cdot v_\lambda$ , and  $L(\lambda)$  is the unique simple top  $W(\lambda)/\text{rad } W(\lambda)$  of  $W(\lambda)$ . Thus, by investigating the weights in  $L(\lambda)$ , we have (i) and (ii).

We prove (iii). We denote by  $\bar{v}_\lambda$  the image of  $v_\lambda$  under the natural surjection  $W(\lambda) \rightarrow L(\lambda)$ . Then, we have  $L(\lambda) = \mathcal{S}_{n,r}^- \cdot \bar{v}_\lambda$ . One sees that

$$M(\lambda) = \bigoplus_{\substack{\mu \in \Lambda_{n,r}^+(\mathbf{m}) \\ \zeta(\lambda) \not\geq \zeta(\mu)}} L(\lambda)_\mu$$

is a  $U_q(\mathfrak{g})$ -submodule of  $L(\lambda)$  since  $\zeta(\mu \pm \alpha_{(i,k)}) = \zeta(\mu)$  for any  $(i,k) \in \Gamma'_{\mathfrak{g}}(\mathbf{m})$ . It is clear that  $M(\lambda)$  is also an  $\mathcal{S}_{n,r}^-$ -submodule of  $L(\lambda)$ , and  $L(\lambda)/M(\lambda) = \mathcal{S}_{n,r}^- \cdot (\bar{v}_\lambda + M(\lambda))$ . For  $F_{(i_1,k_1)} F_{(i_2,k_2)} \cdots F_{(i_l,k_l)} \in \mathcal{S}_{n,r}^-$ , if  $i_j = m_{k_j}$  for some  $j$ , one sees that  $F_{(i_1,k_1)} \cdots F_{(i_l,k_l)} \cdot \bar{v}_\lambda \in M(\lambda)$ . This implies that  $L(\lambda)/M(\lambda)$  is generated by  $\bar{v}_\lambda + M(\lambda)$  as a  $U_q(\mathfrak{g})$ -module, namely we have  $L(\lambda)/M(\lambda) = U_q(\mathfrak{g}) \cdot (\bar{v}_\lambda + M(\lambda))$ . Hence, we have the surjective homomorphism of  $U_q(\mathfrak{g})$ -modules  $\psi : L(\lambda)/M(\lambda) \rightarrow L(\lambda^{(1)}) \boxtimes \cdots \boxtimes L(\lambda^{(r)})$  such that  $\bar{v}_\lambda + M(\lambda) \mapsto \bar{v}_{\lambda^{(1)}} \boxtimes \cdots \boxtimes \bar{v}_{\lambda^{(r)}}$ , where  $\bar{v}_{\lambda^{(k)}}$  is a highest weight vector of  $L(\lambda^{(k)})$  with the highest weight  $\lambda^{(k)}$ . We claim that  $\psi$  is an isomorphism. If  $\psi$  is not an isomorphism, there exists an element  $x \in L(\lambda)_\mu$  such that  $\lambda \neq \mu \in \Lambda_{n,r}^+(\mathbf{m})$ ,  $\zeta(\mu) = \zeta(\lambda)$  and  $e_{(i,k)} \cdot x \in M(\lambda)$  for any  $(i,k) \in \Gamma'_{\mathfrak{g}}(\mathbf{m})$ , namely  $x + M(\lambda) \in L(\lambda)/M(\lambda)$  is a highest weight vector of highest weight  $\mu$  as a  $U_q(\mathfrak{g})$ -module. On the other hand, we have  $E_{(m_k,k)} \cdot x = 0$  for  $k = 1, \dots, r-1$  since  $\zeta(\mu + \alpha_{(m_k,k)}) \succ \zeta(\mu) = \zeta(\lambda)$ . Thus, we have that  $E_{(i,k)} \cdot x \in M(\lambda)$  for any  $(i,k) \in \Gamma'(\mathbf{m})$ . This implies that  $\mathcal{S}_{n,r} \cdot x$  is a proper  $\mathcal{S}_{n,r}$ -submodule of  $L(\lambda)$  which contradict to the irreducibility of  $L(\lambda)$  as an  $\mathcal{S}_{n,r}$ -module. Hence,  $\psi$  is an isomorphism. Then, the isomorphism  $L(\lambda)/M(\lambda) \cong L(\lambda^{(1)}) \boxtimes \cdots \boxtimes L(\lambda^{(r)})$  together with the definition of  $M(\lambda)$  implies (iii).  $\square$

**5.3.** For an algebra  $\mathcal{A}$ , let  $\mathcal{A}\text{-mod}$  be the category of finitely generated  $\mathcal{A}$ -modules, and  $K_0(\mathcal{A}\text{-mod})$  be the Grothendieck group of  $\mathcal{A}\text{-mod}$ . For  $M \in \mathcal{A}\text{-mod}$ , we denote by  $[M]$  the image of  $M$  in  $K_0(\mathcal{A}\text{-mod})$ .

**5.4.** For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$ , let  $d_{\lambda\mu} = [W(\lambda) : L(\mu)]_{\mathcal{S}_{n,r}}$  be the multiplicity of  $L(\mu)$  in the composition series of  $W(\lambda)$  as  $\mathcal{S}_{n,r}$ -modules, and  $\bar{d}_{\lambda\mu} = [W(\lambda^{(1)}) \boxtimes \cdots \boxtimes W(\lambda^{(r)}) : L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})]_{U_q(\mathfrak{g})}$  be the multiplicity of  $L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})$  in the

composition series of  $W(\lambda^{(1)}) \boxtimes \cdots \boxtimes W(\lambda^{(r)})$  as  $U_q(\mathfrak{g})$ -modules. Put

$$\begin{aligned} D &= (d_{\lambda\mu})_{\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})}, & \overline{D} &= (\overline{d}_{\lambda\mu})_{\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})}, \\ X &= (x_{\lambda\mu})_{\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})}, & B &= (\beta_{\lambda\mu})_{\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})}. \end{aligned}$$

Then the decomposition matrix  $D$  of  $\mathcal{S}_{n,r}$  is factorized as follows.

**Theorem 5.5.** *We have that  $B \cdot \overline{D} = D \cdot X$ .*

*Proof.* By the definitions, for  $\lambda \in \Lambda_{n,r}^+(\mathbf{m})$ , we have

$$\begin{aligned} [W(\lambda)] &= \sum_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} d_{\lambda\mu} [L(\mu)] \\ &= \sum_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} d_{\lambda\mu} \left( \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} x_{\mu\nu} [L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})] \right) \\ &= \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \left( \sum_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} d_{\lambda\mu} x_{\mu\nu} \right) [L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})] \end{aligned}$$

in  $\mathcal{K}_0(U_q(\mathfrak{g})\text{-mod})$ . On the other hand, by taking a suitable modular system for  $\mathcal{S}_{n,r}$ , we have

$$\begin{aligned} [W(\lambda)] &= \sum_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\mu} [W(\mu^{(1)}) \boxtimes \cdots \boxtimes W(\mu^{(r)})] \\ &= \sum_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\mu} \left( \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \overline{d}_{\mu\nu} [L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})] \right) \\ &= \sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \left( \sum_{\mu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\mu} \overline{d}_{\mu\nu} \right) [L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})] \end{aligned}$$

in  $K_0(U_q(\mathfrak{g})\text{-mod})$ . By comparing the coefficients of  $[L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})]$ , we obtain the claim of the theorem.  $\square$

As a corollary of Theorem 5.5, we have the product formula for decomposition numbers of  $\mathcal{S}_{n,r}$  which has already obtained by [Saw] in another method.

**Corollary 5.6.** *For  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta(\lambda) = \zeta(\mu)$ , we have*

$$d_{\lambda\mu} = \overline{d}_{\lambda\mu} = \prod_{k=1}^r d_{\lambda^{(k)}\mu^{(k)}},$$

where  $d_{\lambda^{(k)}\mu^{(k)}} = [W(\lambda^{(k)}) : L(\mu^{(k)})]$  is the decomposition number of  $U_q(\mathfrak{gl}_{m_k})$ .

*Proof.* By Lemma 2.6 (ii), for  $\lambda, \mu, \nu \in \Lambda_{n,r}^+(\mathbf{m})$ , if  $\beta_{\lambda\nu}\bar{d}_{\nu\mu} \neq 0$ , then we have  $\lambda \geq \nu \geq \mu$ . Thus, if  $\zeta(\lambda) = \zeta(\mu)$ , we have

$$\sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\nu}\bar{d}_{\nu\mu} = \sum_{\substack{\nu \in \Lambda_{n,r}^+(\mathbf{m}) \\ \zeta(\lambda) = \zeta(\nu) = \zeta(\mu)}} \beta_{\lambda\nu}\bar{d}_{\nu\mu} = \bar{d}_{\lambda\mu},$$

where the last equation follows from Lemma 2.6 (i) and (iii). Similarly, by using Lemma 5.2, we see that  $\sum_{\nu \in \Lambda_{n,r}^+(\mathbf{m})} d_{\lambda\nu}x_{\nu\mu} = d_{\lambda\mu}$ . Hence, Theorem 5.5 implies the claim of the corollary.  $\square$

**Remark 5.7.** In [SW], we also obtained the product formulas for decomposition numbers of  $\mathcal{S}_{n,r}$  which are natural generalization of one in [Saw] as follows. Take  $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}_{>0}^g$  such that  $r_1 + \dots + r_g = r$  as in 3.4. Then, for  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta^{\mathbf{p}}(\lambda) = \zeta^{\mathbf{p}}(\mu)$ , we have

$$(5.7.1) \quad d_{\lambda\mu} = \prod_{k=1}^g d_{\lambda^{[k]\mathbf{p}}\mu^{[k]\mathbf{p}}}$$

by [SW, Theorem 4.17], where  $d_{\lambda^{[k]\mathbf{p}}\mu^{[k]\mathbf{p}}}$  is the decomposition number of  $\mathcal{S}_{n_k, r_k}$  ( $n_k = |\lambda^{[k]\mathbf{p}}|$ ) with parameters  $q, Q_{p_k+1}, \dots, Q_{p_k+r_k}$ . However, the formula (5.7.1) for general  $\mathbf{p} (\neq (1, \dots, 1))$  is not obtained in a similar way as in Corollary 5.6 since  $\overline{\mathcal{S}}_{n,r}^{\mathbf{p}}$  does not realize as a subalgebra of  $\mathcal{S}_{n,r}$  in a similar way as in Lemma 2.2, where  $\overline{\mathcal{S}}_{n,r}^{\mathbf{p}}$  is a subquotient algebra of  $\mathcal{S}_{n,r}$  defined in [SW, 2.12]. (Note that  $\overline{\mathcal{S}}_{n,r}^{\mathbf{p}} \cong \bigoplus_{\substack{(n_1, \dots, n_g) \\ n_1 + \dots + n_g = n}} \mathcal{S}_{n_1, r_1} \otimes \dots \otimes \mathcal{S}_{n_g, r_g}$  by [SW, Theorem 4.15]. Thus, if  $\mathbf{p} = (1, \dots, 1)$ ,  $\overline{\mathcal{S}}_{n,r}^{\mathbf{p}}$  coincides with the right-hand side of the isomorphism in Lemma 2.2.) Hence, in order to obtain the formula (5.7.1) for general  $\mathbf{p}$ , it is essential to take the subquotient algebra  $\overline{\mathcal{S}}_{n,r}^{\mathbf{p}}$  as in [SW].

For special parameters, we see that the matrix  $X$  becomes the identity matrix as the following corollary.

**Corollary 5.8.**

- (i) If  $Q_1 = Q_2 = \dots = Q_r = 0$ , the matrix  $X$  is the identity matrix. In particular, we have  $D = B \cdot \overline{D}$ .
- (ii) If  $q = 1$ ,  $Q_1 = Q_2 = \dots = Q_r$  (not necessary to be 0), the matrix  $X$  is the identity matrix. Moreover, we have  $D = B$  if  $\text{char } F = 0$ .

*Proof.* Assume that  $Q_1 = Q_2 = \dots = Q_r = 0$ . We denote by  $E_{(i,k)}^{(c)} = 1 \otimes E_{(i,k)}^c / [c]!$  (resp.  $F_{(i,k)}^{(c)} = 1 \otimes F_{(i,k)}^c / [c]!$ )  $\in F \otimes_{\mathcal{A}} \mathcal{A}\mathcal{S}_{n,r} \cong {}_F\mathcal{S}_{n,r}$ . By the triangular decomposition of  $\mathcal{S}_{n,r}$ , we have

$$\sigma_{(i,k)}^\lambda = \sum r_{(i_1, k_1, c_1), \dots, (i_l, k_l, c_l)}^{(i'_1, k'_1, c'_1), \dots, (i'_l, k'_l, c'_l)} F_{(i'_1, k'_1)}^{(c'_1)} \cdots F_{(i'_l, k'_l)}^{(c'_l)} E_{(i_1, k_1)}^{(c_1)} \cdots E_{(i_l, k_l)}^{(c_l)} 1_\lambda,$$

for some  $r_{(i_1, k_1, c_1), \dots, (i_l, k_l, c_l)}^{(i'_1, k'_1, c'_1), \dots, (i'_l, k'_l, c'_l)} \in F$ . First, we prove the following claim.

**Claim A:** If  $r_{(i_1, k_1, c_1), \dots, (i_l, k_l, c_l)}^{(i'_1, k'_1, c'_1), \dots, (i'_l, k'_l, c'_l)} \neq 0$  and  $F_{(i'_1, k'_1)}^{(c'_1)} \cdots F_{(i'_l, k'_l)}^{(c'_l)} E_{(i_1, k_1)}^{(c_1)} \cdots E_{(i_l, k_l)}^{(c_l)} 1_\lambda \neq 0$ , then we have  $\zeta(\lambda + c_1 \alpha_{(i_1, k_1)} + \cdots + c_l \alpha_{(i_l, k_l)}) \succ \zeta(\lambda)$ .

Note that  $Q_1 = Q_2 = \cdots = Q_r = 0$ , we see easily that  $\mathcal{H}_{n,r}$  is a  $\mathbb{Z}/r\mathbb{Z}$ -graded algebra with  $\deg(T_0) = \bar{1}$  and  $\deg(T_i) = \bar{0}$ , where we put  $\bar{k} = k + r\mathbb{Z} \in \mathbb{Z}/r\mathbb{Z}$  for  $k \in \mathbb{Z}$ . We can also check that  $m_\lambda$  ( $\lambda \in \Lambda_{n,r}(\mathbf{m})$ ) is a homogeneous element of  $\mathcal{H}_{n,r}$ . Since  $\sigma_{(i,k)}^\lambda(m_\lambda) = m_\lambda \cdot (L_{N+1} + \cdots + L_{N+\lambda_i^{(k)}})$  ( $N = \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{j=1}^{i-1} \mu_j^{(k)}$ ), we have  $\deg(\sigma_{(i,k)}^\lambda(m_\lambda)) = \deg(m_\lambda) + \bar{1}$ . On the other hand, by [W, Lemma 6.10], we see that  $F_{(i'_1, k'_1)}^{(c'_1)} \cdots F_{(i'_l, k'_l)}^{(c'_l)} E_{(i_1, k_1)}^{(c_1)} \cdots E_{(i_l, k_l)}^{(c_l)} 1_\lambda(m_\lambda)$  is a homogeneous element of  $\mathcal{H}_{n,r}$  with degree  $\deg(m_\lambda)$  if  $i_j \neq m_{k_j}$  and  $i'_j \neq m_{k'_j}$  for any  $j = 1, \dots, l$ . Thus, if  $r_{(i_1, k_1, c_1), \dots, (i_l, k_l, c_l)}^{(i'_1, k'_1, c'_1), \dots, (i'_l, k'_l, c'_l)} \neq 0$  and  $F_{(i'_1, k'_1)}^{(c'_1)} \cdots F_{(i'_l, k'_l)}^{(c'_l)} E_{(i_1, k_1)}^{(c_1)} \cdots E_{(i_l, k_l)}^{(c_l)} 1_\lambda \neq 0$ , then there exists  $j$  such that  $i_j = m_{k_j}$ , and this implies that  $\zeta(\lambda + c_1 \alpha_{(i_1, k_1)} + \cdots + c_l \alpha_{(i_l, k_l)}) \succ \zeta(\lambda)$ . Now, we proved Claim A.

We have already shown that  $x_{\lambda\lambda} = 1$ , and  $x_{\lambda\mu} = 0$  for  $\lambda \neq \mu$  such that  $\zeta(\lambda) = \zeta(\mu)$  in Lemma 5.2. Thus, it is enough to show that  $x_{\lambda\mu} = 0$  for  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta(\lambda) \neq \zeta(\mu)$ .

Suppose that  $x_{\lambda\mu} \neq 0$  for some  $\lambda, \mu \in \Lambda_{n,r}^+(\mathbf{m})$  such that  $\zeta(\lambda) \neq \zeta(\mu)$ . We recall that  $L(\lambda) = \mathcal{S}_{n,r}^- \cdot \bar{v}_\lambda$ , where  $\bar{v}_\lambda = v_\lambda + \text{rad } W(\lambda) \in W(\lambda)/\text{rad } W(\lambda) \cong L(\lambda)$ . Then, it is clear that  $L(\lambda)_\mu \neq 0$ . This implies the existence of a non-zero element

$$v' = \sum r_{(i_1, k_1), \dots, (i_c, k_c)} F_{(m_{k'}, k')} F_{(i_1, k_1)} \cdots F_{(i_c, k_c)} \cdot \bar{v}_\lambda \in L(\lambda) \quad (r_{(i_1, k_1), \dots, (i_c, k_c)} \in F)$$

such that  $E_{(i,k)} \cdot v' = 0$  for any  $(i, k) \in \Gamma'_g(\mathbf{m})$ , where the summation runs

$$\{((i_1, k_1), \dots, (i_c, k_c)) \in (\Gamma'_g(\mathbf{m}))^c \mid \alpha_{(i_1, k_1)} + \cdots + \alpha_{(i_c, k_c)} = \alpha\}$$

for some  $\alpha \in \bigoplus_{(i,k) \in \Gamma'_g(\mathbf{m})} \mathbb{Z} \alpha_{(i,k)}$ . Namely  $v'$  is a  $U_q(\mathfrak{g})$ -highest weight vector of highest weight  $\mu' = \lambda - \alpha - \alpha_{(m_{k'}, k')}$ . It is clear that  $\zeta(\lambda) = \zeta(\lambda - \alpha)$ . Since  $E_{(m_k, k)}$  ( $k \neq k'$ ) commute with  $F_{(m_{k'}, k')}$  and  $F_{(j,l)}$  ( $(j, l) \in \Gamma'_g(\mathbf{m})$ ), we have that  $E_{(m_k, k)} \cdot v' = 0$  for any  $k \in \{1, \dots, r-1\} \setminus \{k'\}$ . On the other hand, for  $((i_1, k_1), \dots, (i_c, k_c)) \in (\Gamma'_g(\mathbf{m}))^c$  such that  $\alpha_{(i_1, k_1)} + \cdots + \alpha_{(i_c, k_c)} = \alpha$ , we have

(5.8.1)

$$\begin{aligned} & E_{(m_{k'}, k')} F_{(m_{k'}, k')} F_{(i_1, k_1)} \cdots F_{(i_c, k_c)} \cdot \bar{v}_\lambda \\ &= \left\{ F_{(m_{k'}, k')} E_{(m_{k'}, k')} + \left( q^{(\lambda - \alpha)_{m_{k'}}^{(k')}} - (\lambda - \alpha)_1^{(k'+1)} (q^{-1} \sigma_{(m_{k'}, k')}^{\lambda - \alpha} - q \sigma_{(1, k'+1)}^{\lambda - \alpha}) 1_{\lambda - \alpha} \right) \right\} \\ & \quad 1_{\lambda - \alpha} F_{(i_1, k_1)} \cdots F_{(i_c, k_c)} \cdot \bar{v}_\lambda. \end{aligned}$$

Note that  $\zeta(\lambda - \alpha) = \zeta(\lambda)$ , (5.8.1) together with Claim A implies that

$$E_{(m_{k'}, k')} F_{(m_{k'}, k')} F_{(i_1, k_1)} \cdots F_{(i_c, k_c)} \cdot \bar{v}_\lambda = 0.$$

Thus, we have

$$E_{(m_{k'}, k')} \cdot v' = \sum_{r_{(i_1, k_1), \dots, (i_c, k_c)}} E_{(m_{k'}, k')} F_{(m_{k'}, k')} F_{(i_1, k_1)} \cdots F_{(i_c, k_c)} \cdot \bar{v}_\lambda = 0.$$

As a consequence, we have that  $E_{(i, k)} \cdot v' = 0$  for any  $(i, k) \in \Gamma'(\mathbf{m})$ , and this implies that  $\mathcal{S}_{n, r} \cdot v'$  is a proper  $\mathcal{S}_{n, r}$ -submodule of  $L(\lambda)$ . However, this contradicts to the irreducibility of  $L(\lambda)$  as  $\mathcal{S}_{n, r}$ -module. Thus, we have that  $x_{\lambda\mu} = 0$  for  $\lambda, \mu \in \Lambda_{n, r}^+(\mathbf{m})$  such that  $\zeta(\lambda) \neq \zeta(\mu)$ . Now we proved (i).

Next we prove (ii). Let  $\mathcal{H}_{n, r}$  (resp.  $\mathcal{H}'_{n, r}$ ) be the Ariki-Koike algebra over  $F$  with parameters  $q = 1, Q_1 = \cdots = Q_r = 0$  (resp.  $q = 1, Q'_1 = \cdots = Q'_r = Q' \neq 0$ ), and  $\mathcal{S}_{n, r}$  (resp.  $\mathcal{S}'_{n, r}$ ) be the cyclotomic  $q$ -Schur algebra associated to  $\mathcal{H}_{n, r}$  (resp.  $\mathcal{H}'_{n, r}$ ). We denote by  $T_0, T_1, \dots, T_{n-1}$  (resp.  $T'_0, T'_1, \dots, T'_{n-1}$ ) the generators of  $\mathcal{H}_{n, r}$  (resp.  $\mathcal{H}'_{n, r}$ ) as in 1.1. Then we can check that there exists an isomorphism  $\phi: \mathcal{H}_{n, r} \rightarrow \mathcal{H}'_{n, r}$  such that  $\phi(T_0) = T'_0 - Q'$  and  $\phi(T_i) = T'_i$  ( $1 \leq i \leq n-1$ ). We can also check that  $M^\mu \cong M'^\mu$  for  $\mu \in \Lambda_{n, r}(\mathbf{m})$  under the isomorphism  $\phi$ , where  $M^\mu$  (resp.  $M'^\mu$ ) is the right  $\mathcal{H}_{n, r}$ -module (resp.  $\mathcal{H}'_{n, r}$ -module) defined in 1.3. Thus, we have  $\mathcal{S}_{n, r} \cong \mathcal{S}'_{n, r}$  as algebras. Then (i) implies (ii) since  $\bar{D}$  is the identity matrix when  $q = 1$  if  $\text{char } F = 0$ .  $\square$

### Remarks 5.9.

- (i) In Theorem 5.5, the matrix  $B \cdot \bar{D}$  does not depend on the choice of parameters  $Q_1, \dots, Q_r$ .
- (ii) If  $\mathcal{S}_{n, r}$  is semi-simple, both of  $D$  and  $\bar{D}$  are identity matrices. Thus, we have  $B = X$ .
- (iii) By Theorem 5.5, for  $\lambda, \mu \in \Lambda_{n, r}^+$ , we have

$$d_{\lambda\mu} + x_{\lambda\mu} = \sum_{\nu \in \Lambda_{n, r}^+} \beta_{\lambda\nu} \bar{d}_{\nu\mu} - \sum_{\substack{\nu \in \Lambda_{n, r}^+ \\ \lambda > \nu > \mu}} d_{\lambda\nu} x_{\nu\mu}.$$

Thus, we see that the matrix  $B \cdot \bar{D}$  gives an upper bound of both  $d_{\lambda\mu}$  and  $x_{\lambda\mu}$ .

## § 6. THE ARIKI-KOIKE ALGEBRA AS A SUBALGEBRA OF $\mathcal{S}_{n, r}$

In this section, we consider the algebras over a commutative ring  $R$  with parameters  $q, Q_1, \dots, Q_r \in R$  such that  $q$  is invertible in  $R$ . Hence, we omit the subscript  $R$  for the objects over  $R$ .

**6.1.** For  $\mu \in \Lambda_{n, r}(\mathbf{m})$ , put

$$\begin{aligned} X_{\mu + \alpha_{(i, k)}}^\mu &= \{1, s_{N+1}, (s_{N+1}, s_{N+2}), \dots, (s_{N+1} s_{N+2} \cdots s_{N+\mu_{i+1}^{(k)}-1})\}, \\ X_{\mu - \alpha_{(i, k)}}^\mu &= \{1, s_{N-1}, (s_{N-1}, s_{N-2}), \dots, (s_{N-1} s_{N-2} \cdots s_{N-\mu_i^{(k)}+1})\}, \end{aligned}$$

where  $s_j = (j, j+1) \in \mathfrak{S}_n$  is the adjacent transposition, and  $N = \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{j=1}^i \mu_j^{(k)}$ . Then, by [W, Lemma 6.10, Proposition 7.7 and Theorem 7.16 (i)], we

have

$$(6.1.1) \quad 1_\nu(m_\mu) = \delta_{\mu,\nu} m_\mu,$$

$$(6.1.2) \quad e_{(i,k)}(m_\mu) = q^{-\mu_{i+1}^{(k)}+1} m_{\mu+\alpha_{(i,k)}} \left( \sum_{y \in X_{\mu+\alpha_{(i,k)}}^\mu} q^{\ell(y)} T_y \right),$$

$$(6.1.3) \quad f_{(i,k)}(m_\mu) = q^{-\mu_i^{(k)}+1} m_{\mu-\alpha_{(i,k)}} h_{-(i,k)}^\mu \left( \sum_{x \in X_{\mu-\alpha_{(i,k)}}^\mu} q^{\ell(x)} T_x \right),$$

where  $h_{-(i,k)}^\mu = \begin{cases} 1 & (i \neq m_k), \\ L_N - Q_{k+1} & (i = m_k) \end{cases}$  ( $N := |\mu^{(1)}| + \dots + |\mu^{(k)}|$ ).

**6.2.** Put  $\omega = (\emptyset, \dots, \emptyset, (1^n)) \in \Lambda_{n,r}^+(\mathbf{m})$ . Then, it is well known that  $M^\omega \cong \mathcal{H}_{n,r}$  as right  $\mathcal{H}_{n,r}$ -modules, and that  $1_\omega \mathcal{S}_{n,r} 1_\omega = \text{End}_{\mathcal{H}_{n,r}}(M^\omega, M^\omega) \cong \mathcal{H}_{n,r}$  as  $R$ -algebras. Put  $C_0 = 1_\omega f_{(m_{r-1}, r-1)} e_{(m_{r-1}, r-1)} 1_\omega$ ,  $C_i = 1_\omega f_{(i,r)} e_{(i,r)} 1_\omega \in \mathcal{S}_{n,r}$  for  $i = 1, \dots, n-1$ . Then, we can realize  $\mathcal{H}_{n,r}$  as a subalgebra of  $\mathcal{S}_{n,r}$  as the following proposition.

**Proposition 6.3.**

- (i) *The subalgebra of  $\mathcal{S}_{n,r}$  generated by  $C_0, C_1, \dots, C_{n-1}$  is isomorphic to the Ariki-Koike algebra  $\mathcal{H}_{n,r}$ . Moreover, the subalgebra of  $\mathcal{S}_{n,r}$  generated by  $C_1, \dots, C_{n-1}$  is isomorphic to the Iwahori-Hecke algebra  $\mathcal{H}_n$  of symmetric group  $\mathfrak{S}_n$ .*
- (ii) *Under the isomorphism  $1_\omega \mathcal{S}_{n,r} 1_\omega \cong \mathcal{H}_{n,r}$ , we have  $T_0 = C_0 + Q_r 1_\omega$ ,  $T_i = C_i - q^{-1} 1_\omega$ .*

*Proof.* It is clear that  $C_0, C_1, \dots, C_{n-1}$  are elements of  $1_\omega \mathcal{S}_{n,r} 1_\omega$ . We remark that the isomorphism  $\text{End}_{\mathcal{H}_{n,r}}(M^\omega, M^\omega) \cong \mathcal{H}_{n,r}$  is given by  $\varphi \mapsto \varphi(m_\omega)$  (note that  $m_\omega = 1$ ). Moreover, by (6.1.1) - (6.1.3), we have

$$\begin{aligned} C_0(m_\omega) &= 1_\omega f_{(m_{r-1}, r-1)} e_{(m_{r-1}, r-1)} 1_\omega(m_\omega) \\ &= m_\omega(L_1 - Q_r). \end{aligned}$$

Since  $m_\omega = 1$  and  $L_1 = T_0$ , we have  $C_0(m_\omega) = T_0 - Q_r$ . Similarly, we have  $C_i(m_\omega) = T_i + q^{-1}$  for  $i = 1, \dots, n-1$ . Thus,  $\mathcal{H}_{n,r}$  is generated by  $C_0, C_1, \dots, C_{n-1}$  under the isomorphism  $1_\omega \mathcal{S}_{n,r} 1_\omega \cong \mathcal{H}_{n,r}$ , and  $\mathcal{H}_n$  is generated by  $C_1, \dots, C_{n-1}$ . Now, (ii) is clear.  $\square$

**6.4.** Let  $\mathcal{F} = \text{Hom}_{\mathcal{S}_{n,r}}(\mathcal{S}_{n,r} 1_\omega, -) : \mathcal{S}_{n,r}\text{-mod} \rightarrow \mathcal{H}_{n,r}\text{-mod}$  be the Schur functor. It is well known that, for  $M \in \mathcal{S}_{n,r}\text{-mod}$ ,  $\mathcal{F}(M) = 1_\omega M$  under the isomorphism  $1_\omega \mathcal{S}_{n,r} 1_\omega \cong \mathcal{H}_{n,r}$ . It is also known that  $\{1_\omega L(\lambda) \neq 0 \mid \lambda \in \Lambda_{n,r}^+(\mathbf{m})\}$  gives a complete set of non-isomorphic simple  $\mathcal{H}_{n,r}$ -modules when  $R$  is a field.

Let  $e$  be the smallest positive integer such that  $1 + (q^2) + (q^2)^2 + \dots + (q^2)^{e-1} = 0$ . We say that a partition (not multi-partition)  $\lambda = (\lambda_1, \lambda_2, \dots)$  is  $e$ -restricted if  $\lambda_i - \lambda_{i+1} < e$  for any  $i \geq 1$ .



As a corollary of Corollary 5.8, we have the following classification of simple  $\mathcal{H}_{n,r}$ -modules for special parameters. We remark that this classification has already proved by [AM, Theorem 1.6] and [M1, Theorem 3.7] in other methods.

**Corollary 6.5.** *Assume that  $R$  is a field, and that either  $Q_1 = Q_2 = \cdots = Q_r = 0$  or  $q = 1$ ,  $Q_1 = Q_2 = \cdots = Q_r$ . Then  $1_\omega L(\lambda) \neq 0$  if and only if  $\lambda^{(k)} = \emptyset$  for  $k < r$  and  $\lambda^{(r)}$  is an  $e$ -restricted partition.*

*Proof.* By Corollary 5.8, we have that  $1_\mu L(\lambda) \neq 0$  only if  $\zeta(\mu) = \zeta(\lambda)$ . In particular, we have that  $\lambda^{(k)} = 0$  for any  $k < r$  if  $1_\omega L(\lambda) \neq 0$ . On the other hand,  $L(\lambda) \cong L(\lambda^{(1)}) \boxtimes \cdots \boxtimes L(\lambda^{(r)})$  as  $U_q(\mathfrak{g})$ -modules by Corollary 5.8. In particular, when  $\lambda^{(k)} = \emptyset$  for any  $k < r$ , we have that  $L(\lambda) \cong L(\lambda^{(r)})$  as  $U_q(\mathfrak{gl}_{m_r})$ -modules. Moreover, it is well known that  $1_\omega L(\lambda^{(r)}) \neq 0$  if and only if  $\lambda^{(r)}$  is an  $e$ -restricted partition ([DJ, Theorem 6.3, 6.8]). These results imply the corollary.  $\square$

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